TEERTHANKER MAHAVEER UNIVERSITY
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CENTRE FOR ONLINE & DISTANCE LEARNING

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Programme: Bachelor of Commerce

Course: Operation Research

Course Code: BCPSE402

Semester-IV

Syllabus

OBJECTIVE AND EXPECTED OUTCOME OF THE COURSE:

The objective of the course is to acquaint the students with the applications of the operations research to business and industry and help them to grasp the significance of analytical approach to decision making.

Unit I

Operations Research: Nature, Scope, Methodology of operations research and role in managerial decision making. **Linear Programming:** Formulation of problem and its solution by Graphical and Simplex methods, Duality.

Unit II

Transportation Problems: Formulation, Optimal solution, Unbalanced Transportation problem, Degeneracy. **Assignment Problems:** Formulation, Optimal solution, Variants of Assignment Problems, Travelling Salesman problems

Unit III

Game Theory: Games with pure and mixed strategies, Saddle Point, Odds method, Principle of dominance, Sub Games method, Equal gains method and LPP- Graphic Method **Sequencing Problems**: Processing n jobs through two machines, Processing in jobs through three machines.

Unit IV

Inventory Models – EOQ Models, Quantity Discount Models, Purchase inventory models with One Price Break (Single Discount) and Multiple Discount breaks. **Network Analysis**: PERT and CPM Model, difference between PERT and CPM, Computation of Critical Path, Slack, Floats and Probability of project completion by a target date.

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Chapter 1: Introduction to Operations Research

1.0 OBJECTIVES

The students should be able to understand:

- Meaning of Operation Research
- Different aspects of operation research
- Application of operation research in Managerial decision making

1.1 INTRODUCTION and Meaning of Opeartion Research

Operation Research is generally accepted that the field originated in England during World War II. The impetus for its origin was the development of radar defense systems for the Royal Air Force, and the first recorded use of the term Operations Research is attributed to a British Air Ministry official named A. P. Rowe who constituted teams to do "operational researches" on the communication system and the control room at a British radar station. The studies had to do with improving the operational efficiency of systems (an objective which is still one of the cornerstones of modern O.R.). This new approach of picking an "operational" system and conducting "research" on how to make it run more efficiently soon started to expand into other arenas of the war. Perhaps the most famous of the groups involved in this effort was the one led by a physicist named P. M. S. Blackett which included physiologists, mathematicians, astrophysicists, and even a surveyor. This multifunctional team focus of an operations research project group is one that has carried forward to this day. Blackett's biggest contribution was in convincing the authorities of the need for a scientific approach to manage complex operations, and indeed he is regarded in many circles as the original operations research analyst.O.R. made its way to the United States a few years after it originated in England. Its first presence in the U.S. was through the U.S. Navy's Mine Warfare Operations Research Group; this eventually expanded into the Antisubmarine Warfare Operations Research Group that was led by Phillip Morse, which later became known simply as the Operations Research Group. Like Blackett in Britain, Morse is widely regarded as the "father" of O.R. in the United States, and many of the distinguished scientists and mathematicians that he led went on after the end of the war to become the pioneers of O.R. in the United States.In the years immediately following the end of World War II, O.R. grew rapidly as many scientists realized that the principles that they had applied to solve problems for the military were equally applicable to many problems in the civilian sector. These ranged from short-term problems such as scheduling and inventory control to long-term problems such as strategic planning and resource allocation. George Dantzig, who in 1947 developed the simplex algorithm for Linear Programming (LP), provided the single most important impetus for this growth. To this day, LP remains one of the most widely used of all O.R. techniques and despite the relatively recent development of interior point methods as an alternative approach, the simplex algorithm (with numerous computational refinements) continues to be widely used. The second major impetus for the growth of O.R. was the rapid development of digital computers over the next three decades. The simplex method was implemented on a computer for the first time in 1950, and by 1960 such implementations could solve problems with about 1000 constraints. Today, implementations on powerful workstations can routinely solve problems with hundreds of thousands of variables and constraints. Moreover, the large volumes of data required for such problems can be stored and manipulated very efficiently.Once the simplex method had been invented and used, the development of other methods followed at a rapid pace. The next twenty years witnessed the development of most of the O.R. techniques that are in use today including nonlinear, integer and dynamic programming, computer simulation, PERT/CPM, queuing theory, inventory models, game theory, and sequencing and scheduling algorithms. The scientists who developed these methods came from many fields, most notably mathematics, engineering and economics. It is interesting that the theoretical bases for many of these techniques had been known for years, e.g., the EOQ formula used with many inventory models was developed in 1915 by Harris, and many of the queuing formulae were developed by Erlang in 1917. However, the period from 1950 to 1970 was when these were formally unified into what is considered the standard toolkit for an operations research analyst and successfully applied to problems of industrial significance. The following section describes the approach taken by operations research in order to solve problems and explores how all of these methodologies fit into the O.R. framework.

1.1.1 Meaning of Opeartion Research

Analytical methods used in OR include mathematical logic, simulation, network analysis, queuing theory, and game theory . The process can be broadly broken down into three steps: A set of potential Operations research (OR) are an analytical method of problem-solving and decision-making that is useful in the management of organizations. In operations research, problems are broken down into basic components and then solved in defined steps by mathematical analysis. solutions to a problem is developed, The alternatives derived in the first step are analyzed and reduced to a small set of solutions most likely to prove workable, The alternatives derived in the second step are subjected to simulated implementation and, if possible, tested out in real-world situations. In this final step, psychology and management science often play important roles.

1.1.2 Definition of Operation Research

According to Philip McCord Morse and George E. Kimball, "Operations research is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control."

According to Russell L. Ackoff and C. West Churchman, "Operations research is the application of scientific methods to arrive at the optimal solutions to the problems."

1.2 Nature of Operation Research

The significant features of operations research include the followings:

1. **Decision-making**. Every industrial organization faces multifaceted problems to identify best possible solution to their problems. OR aims to help the executives to obtain optimal solution with the use of OR techniques. It also helps the decision maker to improve his creative and judicious capabilities, analyses and understand the problem situation leading to better control, better co-ordination, better systems and finally better decisions.
2. **Scientific Approach**. OR applies scientific methods, techniques and tools for the purpose of analysis and solution of the complex problems. In this approach there is no place for guess work and the person bias of the decision maker.
3. **Inter-disciplinary Team Approach**. Basically the industrial problems are of complex nature and therefore require a team effort to handle it. This team comprises of scientist/mathematician and technocrats. Who jointly use the OR tools to obtain a optimal solution of the problem. The tries to analyses the cause and effect relationship between various parameters of the problem and evaluates the outcome of various alternative strategies.
4. **System Approach**. The main aim of the system approach is to trace for each proposal all significant and indirect effects on all sub-system on a system and to evaluate each action in terms of effects for the system as a whole. The interrelationship and interaction of each sub-system can be handled with the help of mathematical/analytical models of OR to obtain acceptable solution.
5. **Use of Computers**. The models of OR need lot of computation and therefore, the use of computers becomes necessary. With the use of computers it is possible to handle complex problems requiring large amount of calculations. The objective of the operations research models is to attempt and to locate best or optimal solution under the specified conditions. For the above purpose, it is necessary that a measure of effectiveness has to be defined which must be based on the goals of the organisation. These measures can be used to compare the alternative courses of action taken during the analysis.

1.3 Scope of Operation Research

In modern times OR has entered successfully in many different areas of research. It is useful in the following various important fields

1. Application in agriculture

With the sudden increase of population and resulting shortage of food, every country is facing the problem of

- Optimum allocation of land to a variety of crops as per the climatic conditions
- Optimum distribution of water from numerous resources like canal for irrigation purposes

Hence there is a requirement of determining best policies under the given restrictions. Therefore a good quantity of work can be done in this direction.

2. Application in finance

In these recent times of economic crisis, it has become very essential for every government to do a careful planning for the economic progress of the country. OR techniques can be productively applied

- To determine the profit plan for the company
- To maximize the per capita income with least amount of resources
- To decide on the best replacement policies, etc

3. Application in industry

If the industry manager makes his policies simply on the basis of his past experience and a day approaches when he gets retirement, then a serious loss is encounter ahead of the industry. This heavy loss can be right away compensated through appointing a young specialist of OR techniques in business management. Thus OR is helpful for the industry director in deciding optimum distribution of several limited resources like men, machines, material, etc to reach at the optimum decision.

4. Application in marketing

With the assistance of OR techniques a marketing administrator can decide upon

- Where to allocate the products for sale so that the total cost of transportation is set to be minimum
- The minimum per unit sale price
- The size of the stock to come across with the future demand
- How to choose the best advertising media with respect to cost, time etc?
- How, when and what to buy at the minimum likely cost?

5. Application in personnel management

A personnel manager can utilize OR techniques

- To appoint the highly suitable person on minimum salary
- To know the best age of retirement for the employees
- To find out the number of persons appointed in full time basis when the workload is seasonal

6. Application in production management

A production manager can utilize OR techniques

- To calculate the number and size of the items to be produced
- In scheduling and sequencing the production machines
- In computing the optimum product mix
- To choose, locate and design the sites for the production plan

7. Application in Insurance

OR approach is also applicable to facilitate the insurance offices to decide

- What should be the premium rates for a range of policies?
- How well the profits could be allocated in the cases of with profit policies?

8. Application in Education and Training: OR can be used for obtaining optimum number of schools with their locations, optimum mix of students/teacher student ratio, optimum financial outlay and other relevant information in training of graduates to meet out the national requirements.

9. Application in Transportation: Transportation models of OR can be applied to real life problems to forecast public transport requirements, optimum routing, forecasting of income and expenses, project management for railways, railway network distribution, etc. In the same way it can be useful in the field of communication.

10. Application in Home Management and Budgeting : OR can be effectively used for control of expenses to maximize savings, time management, work study methods for all related works. Investment of surplus budget, appropriate insurance of life and properties and estimate of depreciation and optimum premium of insurance etc.

Application 1.

Enumerate Application of Operation Research in Education , Training and Insurance:

Check Your Progress A

Q1.Operations Research (OR) , which is a very powerful tool for ----------------

Research

Decision – Making

Operations

None of the above

Q2. Who coined the term Operations Research?

J.F. McCloskey

F.N. Trefethen

P.F. Adams

Both A and B

Q3. The term Operations Research was coined in the year --------------

1950

1940

1978

1960

1.5 Role of Operation Research in Managerial Decision Making

The Operation Research may be considered as a tool which is employed to raise the efficiency of management decisions. OR is the objective complement to the subjective feeling of the administrator (decision maker). Scientific method of OR is used to comprehend and explain the phenomena of operating system.

The benefits of OR study approach in business and management decision making may be categorize as follows

1. **Helpful in Better control**

The management of large concerns finds it much expensive to give continuous executive supervisions over routine decisions. An OR approach directs the executives to dedicate their concentration to more pressing matters. For instance, OR approach handles production scheduling and inventory control.

2.Helpful in Better coordination

Sometimes OR has been very helpful in preserving the law and order situation out of disorder. For instance, an OR based planning model turns out to be a vehicle for coordinating marketing decisions with the restrictions forced on manufacturing capabilities.

3.Helpful in Better system

OR study is also initiated to examine a particular problem of decision making like setting up a new warehouse. Later OR approach can be more developed into a system to be employed frequently. As a result the cost of undertaking the first application may get better profits.

4. Helpful in Better decisions

OR models regularly give actions that do enhance an intuitive decision making. Sometimes a situation may be so complex that the human mind can never expect to assimilate all the significant factors without the aid of OR and computer analysis.

1.6 Major models in Operation Research

There are different models used in operation research. These models are derived from Engineering and mathematics and used for different problems solution in operation research:

1. **Linear Programming (L.P.):** Linear programming is basically a constrained optimisation technique which tries to optimise some criterion within some constraints. It consists of an objective function which is some measure of effectiveness like profit, loss or return on investment and several boundary conditions putting restriction on the use of resources. Objective function and boundary conditions are linear in nature. There are methods available to solve a linear programming problem.

2. **Waiting Line or Queuing Theory:** This deals with the situation in which queue is formed or the customers have to wait for service or machines wait for repairmen and therefore concept of a queue is involved. If we assume that there are costs associated with waiting in line, and if there are costs of adding more service facilities, we want to minimize the sum of costs of waiting and the costs of providing service facilities. Waiting line theory helps to

make calculations like number of expected member of people in queue, expected waiting time in the queue, expected idle time for the server, etc. These calculations then can be used to determine the desirable number of service facilities or number of servers.

3. **Game Theory :** It is used for decision-making under conflicting situations where there are one or more opponents. The opponents, in game theory, are called players. The motives of the players are optimizing gains or minimizing losses. The success of one player tends to be at the cost of others and hence they are in conflict. Game theory models, a conflict situation arises and helps to improve the decision process by formulating appropriate strategy.

4. **Inventory Control Models**: These models deal with the quantities which are either to be purchased or stocked since each factor involves cost. The purchase and material managers are normally encounter such situations. Therefore, inventory models provide rational answer to these questions in different situations of supply and demand for different kind of materials. Inventory control models help managers to decide ordering time, reordering level and optimal ordering quantity. The approach is to prepare a mathematical model of the situation that expressed total inventory costs in terms of demand, size of order, possible over or under stocking and other relevant factors and then to determine optimal order size, optimum order level etc. using calculus or some other technique.

5. **Simulation**: It is basically data generating technique, where sometimes it is risky, cumbersome, or time consuming to conduct real study or experiment to know more about situation or problem. The available analytical methods cannot be used in all situations due to large number of variables or large number of interrelationships among the variables and the complexity of relationship; it is not possible to develop an analytical model representing the real situation.

6. **Non-Linear Programming**: These models may be used when either the objective function or some of the constraints are not linear in nature. Non-linearity may be introduced by such factors as discount on price of purchase of large quantities and graduated income tax etc. Linear programming may be employed to approximate the non-linear conditions, but the approximation becomes poorer as the range is extended. Non-linear methods may be used to

determine the approximate area in which a solution lies and linear methods may be used to obtain a more exact solution.

7. **Integer Programming:** This method can be used when one or more of the variables can only take integer values. Examples are the number of trucks in a fleet, the number of generators in a power house and so on. Approximate solutions can be obtained without using integer programming methods, but the approximation generally becomes poorer as the number becomes smaller. There are techniques to obtain solution of integer programming problems.

8. **Dynamic Programming:** This is a method of analyzing multistage decision processes, in which each elementary decision is dependent upon those preceding it as well as upon external factors. It drastically reduces the computational efforts otherwise necessary to analyze results of all possible combinations of elementary decisions.

9. **Sequencing Theory:** This is related to waiting line theory and is applicable when the facilities are fixed, but the order of servicing may be controlled. The scheduling of service or the sequencing of jobs is done to minimize the relevant costs and time.

10. **Markov Process**: It is used for decision-making in situations where various states are defined. The probability of going from one state to another is known and depends on the present state and is independent of how we have arrived at that state. Theory of Markov process helps us to calculate long run probability of being in a particular state (steady state probability), which is used for decision-making.

11. **Network Scheduling—PERT and CPM :** These techniques are used to plan, schedule and monitor large projects such as building construction, maintenance of computer system installation, research and development design etc. The technique aims at minimizing trouble spots, such as, delays, interruptions and production bottlenecks, by identifying critical factors and coordinating various parts of overall job/project. The project/job is diagrammatically represented with the help of network made of arrows representing different activities and interrelationships among them. Such a representation is used for identifying critical activities and critical path. Two basic techniques in network scheduling are Program Evaluation and

Review Technique (PERT) and Critical Path Method (CPM). CPM is used when time taken by activities in a project are known for sure and PERT is used when activities time is not known for sure—only probabilistic estimate of time is available to the users.

12. **Symbolic Logic**: It deals with substituting symbols for words, classes of things or functional systems. Symbolic logic involves rules, algebra of logic and propositions. There have been only limited attempts to apply this technique to business problems; however has had extensive application in the design of computing machinery.

13. **Information Theory**: Information theory is an analytical process transferred from the electrical communications field to operations research. It seeks to evaluate the effectiveness of information flow within a given system. Despite its application mainly to communication networks, it has had a indirect influence in simulating the examination of business organizational structures with a view to improving information or communication flow.

14. **Utility/Value Theory**: It deals with assigning numerical significance to the worth of alternative choices. To date, this has been only a concept and is in the stage of elementary model formulation and experimentation and can be useful in decision-making process.

Application 2

Explain the Role of two players in game theory:

Check Your Progress B

Q1.Operations Research uses models to help the management to determine its ----------- scientifically

(a) Policies
(b) Actions
(c) Both A and B
(d) None of the above

Q2.Operations Research is a -----------------

(a) Science
(b) Art
(c) Mathematics
(d) Both A and B

Q3.What has been constructed for Operations Research problems and methods for solving the models that are available in many cases?

(a) Scientific Models
(b) Algorithms
(c) Mathematical Models
(d) None of the above

1.6 Limitations of Operation Research

There are certain problems in operation research:

1. In the quantitative analysis of operations research, certain assumptions and estimates are made for assigning quantitative values to factors involved. If such estimates are wrong, the result would be- equally misleading.

2. Many management problems do not lend themselves to quantitative measurement and analysis. Intangible factors of any problem concerning human behaviour cannot be quantified

accurately and all the patterns of relationships among the factors may not be covered. Accordingly, the outward appearance of scientific accuracy through the use of numbers and equations becomes unrealistic.

3. The quantitative methods of operations research are many cases costly, elaborate and sophisticated in nature. Although complex problems are fit for analysis by tools of operations research, relatively simple problems have no economic justification for this type of quantitative analysis.

4. A knowledge of some concepts of mathematics and statistics is prerequisite for adoption of quantitative analysis by the managers. According to the present training and experience of most managers, the actual use of these tools may be confined to a few cases.

5. Operations research is not a substitute for the entire process of decision making and it does not relieve the managers of their task of decision making. In one phase of decision making viz., selection of best solution through the evaluation of alternatives, operations research comes into the picture.

Managers have to prepare the ground-work for the introduction of operations research through several steps in decision making viz., diagnosis of problem, analysis of problem and development of alternatives; and even after the selection of best solution by operations research, managers have to put the decision into- effect and to institute a system of follow-up.

1.7 Some practical Examples of Operation Research Problems

1. Scheduling: of aircrews and the fleet for airlines, of vehicles in supply chains, of orders in a factory and of operating theatres in a hospital.
2. Facility planning: computer simulations of airports for the rapid and safe processing of travelers, improving appointments systems for medical practice.
3. Planning and forecasting: identifying possible future developments in telecommunications, deciding how much capacity is needed in a holiday business.
4. Yield management: setting the prices of airline seats and hotel rooms to reflect changing demand and the risk of no shows.

5. Credit scoring: deciding which customers offer the best prospects for credit companies.
6. Marketing: evaluating the value of sale promotions, developing customer profiles and computing the life-time value of a customer.
7. Defence and peace keeping: finding ways to deploy troops rapidly.

1.7 Summary

Operations research (OR) are an analytical method of problem-solving and decision-making that is useful in the management of organizations. In operations research, problems are broken down into basic components and then solved in defined steps by mathematical analysis. Basically the industrial problems are of complex nature and therefore require a team effort to handle it. This team comprises of scientist/mathematician and technocrats. Who jointly use the OR tools to obtain a optimal solution of the problem. There are various models in operation research such as linear programming, game theory, assignment, simulation etc. which are used for problem solving in different areas.

1.9 Glossary

- **Operational Research (OR) :** The application of the methods of science to complex problems arising in the management of large systems of people, machines, materials and money in industry, business, government and community services.
- **Model:** A representation of reality that reproduces the essential features or elements of the system or entity being studied.
- **Parameter:** A value circumscribing a problem that is constant within a process, but may be changed for subsequent processes.
- **Decision variable:** A quantity that varies, or a vector of such quantities, that is under the control of the decision maker.

1.10 Answers to check your progress

Check your progress A

1.b

2.d

3.b

Check Your Progress B

1.C

2.D

3.C

1.11 References/ Suggested Readings

- Hillier, F.S. and Lieberman, J. G., *Operations Research*, Tata McGraw Hill, 2009, 10th Reprint.
- Anderson, D.R., Sweeney D. J. and Williams, T.A., *An Introduction to Management Science: Quantitative Approach to Decision Making*, South Western Cengage Learning, 11th Edition.
- Natarajan, A.M., Balasubramani, P. and Tamilarasi, A., *Operations Research*, Pearson, 2012, 9th Edition.

1.12 Terminal and Model Questions

Q1. What is operations research and explain briefly its applications in industrial organizations?

Q2. What are the characteristics of operations research? Discuss.

Q 3. Discuss the importance of OR in decision-making process.

Q4. Enumerate, with brief description, some of the important techniques used in OR.

Q5. Discuss the limitations of operation research.

Q6. Describe the various steps involved in OR study.

Q7. Discuss significance and scope of operation research.

Q 8. Describe briefly the different phases of operation research.

Q9. Explain steps involved in the solution of OR problems.

Q 10. Illustrate the importance of features in OR.

Chapter 2: LINEAR PROGRAMMING PROBLEM

2.0 OBJECTIVES

The students should be able to understand:

- Importance of linear programming problem (LPP)
- Elements and assumptions of LPP
- Formulation of LPP by using slack, surplus and artificial variable.
- Simplex and Graphical Method of LPP

2.1 INTRODUCTION

Linear programming is an important technique in operations research used for problem solving and decision making. The importance of this technique generates from definition of operations management. The subject of operations management deals with conversion of limited resources into desired outputs in most efficient and productive way. The aspect that needs to be focused is that every industry faces with limitation of limited and finite resources. The challenge is to fulfill the needs of customers by converting these limited resources in a way that firm earns profits and be able to beat its competition. Operations research as a subject and linear programming as a technique is an effective method of meeting this challenge. Linear programming specifically allocates limited resources to diverse activities in such a way to achieve the objective of either minimization or maximization. Elaborating further on limited resources aspect is that every firm or business works under number of constraints. These constraints can be financial, availability of resources such as raw material, manpower, skills etc. linear programming provides a mathematical technique to allocate such resources used to perform competing activities in order to achieve certain decided objectives. Following are certain examples where linear programming can be applied:

- A marketing manager wants to promote launch of new product to the maximum. The manager faces with the problem of allocating limited promotion budget on different media channels. Should product be promoted more online or on TV or through print media? How many limited financial resources are allocated to which media channel?
- An operations manager wants to maximize profits by producing more of two kinds of products. How much raw materials and manpower should be allocated for production of each type of product? What would be the impact on maximization of profits if one product is manufactured more than the other?
- A financial manager has limited corpus of money and he/she would like to invest in three different schemes such as stock markets, government securities and bank deposits in such a way that return are maximum. Linear programming can help financial manager in allocating a fixed amount of money in three instruments to achieve objective of maximization.
- Lastly, an individual wants to limit his/her calorie intake by spending a fixed amount on recommended four diets. Each diet provides a fixed amount of calories in form of proteins, fats and carbohydrates. The individual has to function under constraints of limited amount of money and limited intake of calories from four diets. Linear programming can again be used to find amount of money to be allocated on each diet so that objective of minimization of calorie intake be achieved.

2.2 ELEMENTS OF LINEAR PROGRAMMING MODEL

Following illustration is explained to understand the elements used in construction of linear programming model.

A company is involved in manufacturing of two types of smart phones: A and B. Both are manufactured in same factory and use same machines and workforce. Profits earned from sale of one unit of A and B is Rs. 1000 and Rs.1200 respectively. It has to supply a fixed amount of both types of smart phones in a week which comprises of 80 hours of production time to the maximum. One unit of A takes 2 hours and that of B takes 3 hours for production. Also available storage capacity is limited to 100 square feet. One unit of A takes 0.5 sq.ft. and one unit of B takes 0.7 sq.ft of space. Manger is faced with problem of how many units of each to be produced so as to maximize profits.

Following elements can be deduced from above illustration:

- ***Decision Variables:*** Achievement of maximum profit objective depends on production of certain number of units of A and B. So profit is directly related to *number of units* of products manufactured making it as a deciding variable.
- ***Objective:*** The manager wants to maximize profits by selling certain units of A and B. So, each problem has an objective to fulfill. In this case objective is of *maximization*. In some cases such as transportation cost objective can be of *minimization*.
- ***Constraints:*** Manager has to produce under two constraints: production capacity i.e. existing plant can only work for 80 hours within which it has to fulfill demand and storage space i.e. only limited amount of finished products can be stored in available space. The question is out of 80 hours how much time needs to be devoted for production of A and how much to B. Similarly if A is produced in higher quantity do manager have space to store them. So, how much space should be allocated to A and B. These constraints have direct impact on decision variable of number of units to be produced which would decide objective of maximization.
- ***Coefficients:*** both decision variables and objective function use coefficients to represent level of activity per unit. For instance, production time per unit for product A is 2 hours and for product B is 3 hours. Similarly in objective function of maximization coefficients indicate profit per unit of each type of product.
- ***Inequality:*** Inequalities such as less than (<), less than equal to (<=), greater than (>) or greater than equal to (>=) represents the relation between level of activity and maximum or minimum resources available corresponding to that activity. For instance, above illustration indicates that maximum production hours available are 80 hours so level of activity w.r.t. production hours should be less than or equal to 80 hours. Similarly, other inequalities can be formulated in constraint equations.

Certain symbols are commonly used to denote various elements of linear programming model. These symbols are listed below:

x_j = number of decision variable (for j = 1, 2,….,n)

Z = objective function of maximization or minimization

b_i = constraints indicating maximum or minimum amount of resource available to carry put a particular activity i (for i = 1, 2,….,n)

a_{ij} = coefficients of decision variables indicating amount of resource i that is available for a particular decision variable j

c_j = coefficients w.r.t. each decision variable in the objective function indicating either profit or cost for a particular decision variable

Thus, a general form of linear programming problem would be formulated as:

Objective function of Maximization or minimization

$$Z = c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n$$

Subject to: $a_{11}x_1 + a_{12}x_2 + \ldots\ldots.. + a_{1j}x_j <= b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots\ldots.. + a_{2j}x_j < = b_2$

$\ldots\ldots\ldots. + \ldots\ldots\ldots.. + \ldots\ldots\ldots + \ldots\ldots\ldots. < = \ldots..$

$a_{i1}x_1 + a_{12}x_2 + \ldots\ldots\ldots. + a_{ij}x_j < = b_i$

For above illustration and by using explained symbols mathematical representation of model is:

Maximize $Z = 1000x_1 + 1200x_2$

Subject to following constraints:

Production capacity: $2x_1 + 3x_2 <= 80$

Storage capacity: $0.5x_1 + 0.7x_2 <= 100$

and $x_1, x_2 >= 0$

2.3 ASSUMPTIONS OF LINEAR PROGRAMMING MODEL

A linear programming problem should follow certain assumptions for proper modeling of its activities and the data being modeled. Following are such assumptions:

Proportionality: This assumption is associated with both objective function and constraints. It says that contribution of each decision variable to the objective function is proportional to the level of activity or number of units produced by that activity. For instance if objective function is $Z = 1000x_1 + 1200x_2$ and number of units to be produced were decided as 15 and 10 of x_1 and x_2 respectively, then profit generated by production of one unit of x_1 was Rs.1000 and for production of 15 units of x_1 would be 15,000. Similarly production of one unit of x_2 gave profit of Rs.1200 so production of 10 units of x_2 would give profit of Rs.12, 000. For one of the constraints like production function this assumption can be used to check the feasibility of solution. One unit of x_1 requires 2 hours for production so 15 units would take 30 hours. Similarly one unit of x_2 takes 3 hours for production so 10 units would take 30 hours. Thus, a total of 60 hours are required for production of total of 25 units. As left hand side is less than right hand side as it should be according to inequality of production constraint so solution is feasible. But it also indicates that entire production capacity is not used and there is a slack of 20 hours.

Additivity: This assumption says that every function in linear programming problem whether it is objective function or constraint function is the sum of individual contributions of a particular activity related to each decision variable. For example, if one unit of product A and B takes t1 and t2 hours for production respectively then total time take to produce one unit of A and B would be t1 + t2 hours. Similarly if one unit of product A and B gives a profit of c1 and c2 respectively then total profit of manufacturing one unit of A and B would be c1 + c2.

Divisibility: This assumption says that decision variables in a linear programming problem can take values even non integer values within given feasible region. This implies that these variables are not restricted only to integer values. So far decision variables fulfill constraint equations they can take any values. For instance if objective function is $Z = 1000x_1 + 1200x_2$ and constraint are $2x_1 + 3x_2 <= 80$ and $x_1, x_2 >= 0$ then maximum units of x_1 ranges from zero to 40 and maximum units of x_2 ranges from zero to 26.66. Thus, decision variables can take integer as well as non-integer values.

Certainty: All coefficients also termed as parameters in objective and constraint functions are assumed to be known constant. For instance time required to produce one unit of A and B, storage space required to store one unit of A and B and profits generated by sale of one unit of A and B should be known constant values.

Exercise 1

1. A feasible solution to a linear programming problem

a) Must satisfy all of the problem's constraints simultaneously

b) Need not satisfy all of the constraints, only the non-negativity constraints

c) Must be a corner point of the feasible region

d) Must give the maximum possible profit

2. Which of the following inequalities are applicable in LPP

(a) <= (b) >= (c) = (d) all of the above

3. Decision and constraint functions can take

(a) Integer values only (b) non-integer values only (c) both

2.4 <u>PRODUCT MIX LINEAR PROGRAMMING PROBLEM: AN ILLUSTRATION OF MAXIMIZATION OBJECTIVE</u>

A glass manufacturing company produces variety of glasses used in manufacturing of different types of windows and doors. The company has decided to launch two new products A and B to catch up with competition. Company has three plants (P, Q and R) where existing product line was manufactured and it was decided some production time has to be freed up in these production plants for manufacturing of A and B. the operations research team needed to found data regarding:

- Number of hours of production time used to manufacture each type of product per batch. For instance, if a batch of 20 units need to be manufactured each of A and B then how much time is required to produce these batches.
- Number of hours of production time available per week in each plant to produce a given batch of A and B. For instance, to manufacture a batch of 20 units each of A and B what is the production time available in each plant which can be allocated for manufacturing of certain units of A and B?
- Lastly profit per unit generated from manufacture of each unit of new product per batch.

The team found that product A can use 1 hour of production time from plant P out of maximum available of 4 hours and 3 hours from plant R out of maximum production time available of 18 hours. Product B can utilize 2 hours from plant Q out of maximum production time available of 12 hours and 2 hours from plant R per week for production of decided batch. Profit by selling one unit of product A and B was determined to be Rs.3000 and Rs.5000 respectively. Objective was to find out how many units of A and B should be manufacture so as to achieve maximum profit.

This illustration is solved both by Simplex and Graphical method.

2.4.1 Simplex Method: Simplex method requires following steps for solving of linear programming problem

Step 1: *Formulation of linear programming model*

x_1 = number of units of product A produced per week

x_2 = number of units of product B produced per week

Z = total profit per week from producing A and B (given in thousands)

Maximize $Z = 3x_1 + 5x_2$

Subject to:
$$x_1 \le 4$$
$$2x_2 \le 12$$
$$3x_1 + 2x_2 \le 18$$

and $x_1, x_2 \ge 0$

The inequalities in constraint functions are less than or equal to implying that left hand side is less than or equal to right hand side. Thus, to convert inequality into equality another variable called as ***slack variable*** is added in all three equations. Slack variables indicate idle resource of particular activity and they are symbolized as s_1, s_2 and s_3 for three constraints. These slack variables are termed as *basis variables.*

This results in following linear programming model.

Maximize $Z = 3x_1 + 5x_2 + 0s_1 + 0s_2 + 0s_3$

Subject to:

Plant P: $x_1 + 0x_2 + 1s_1 + 0s_2 + 0s_3 = 4$

Plant Q: $0x_1 + 2x_2 + 0s_1 + 1s_2 + 0s_3 = 12$

Plant R: $3x_1 + 2x_2 + 0s_1 + 0s_2 + 1s_3 = 18$

Table 4.4.1.1 shows the simplex table

Table 4.4.1.1							
Decision Variables		***x_1***	***x_2***	***s_1***	***s_2***	***s_3***	
Coefficients of decision variables in objective function		***3***	***5***	***0***	***0***	***0***	
Basis variables	**Coefficients of basis variables c_b in objective function**	**Coefficients of constraint functions for three plants**					**Maximum available resource**
s_1	0	1	0	1	0	0	***4***
s_2	0	0	2	0	1	0	***12***
s_3	0	3	2	0	0	1	***18***

Step 2: *Find Initial Basic Feasible Solution (ibfs):*

Equations of constraints show that there are five variables: x1 and x2 are decision variables also called as non-basic variables and s1, s2 and s3 are basic variables. To find values for these five variables we have only three equations. So to find a solution first step is put any two variables as zero. This condition is fulfilled by non-negativity criteria i.e. all variables in the objective function should be non-negative. Generally, x1 and x2 are assigned as zero indicating initial situations implying that production is yet to start and in the beginning quantity of two products is zero. So by assigning x1 and x2 as zero we get values of basic variables as s1 = 4, s2 = 12 and s3 = 18. This indicates all resources allocated for manufacturing of A and B in three plants is yet to be used. Also objective function of maximum profits would come out to be zero when values of these variables are used in objective function. So mathematically we are starting from beginning and would try to find a better solution than zero profits.

To find whether solution can be improved: To understand this follow following two steps:

(i) As discussed initial solution indicates start of operations when no units are produced. A better solution means that when production starts or x1 is increased from 0 to 1. So initial profits for each decision variable are represented by z_j and calculated as:

$Z_1 = 0*(1) + 0*(0) + 0*(3) = 0$

$Z_2 = 0*(0) + 0*(2) + 0*(2) = 0$ and so on

All initial profits are shown in table 4.4.1.2 in z_j row. Remember that values in first column represented by x_1 units of product A are multiplied by coefficients of basis variables c_b in objective function.

Table 4.4.1.2

Decision Variables		x_1	x_2	s_1	s_2	s_3	
Coefficients of decision variables in objective function		***3***	***5***	***0***	***0***	***0***	
Basis variables	**Coefficients of basis variables c_b in objective function**	**Coefficients of constraint functions for three plants**					**Maximum available resource**
s_1	0	1	0	1	0	0	***4***
s_2	0	0	2	0	1	0	***12***
s_3	0	3	2	0	0	1	***18***
	z_j	0	0	0	0	0	
Net evaluation row	$c_j - z_j$	3	5	0	0	0	

(ii) $C_j - z_j$ represents the change in profits from zero number of units of a particular product to one unit. It has been stated that with production of one unit of A profit obtained is Rs.3. So net profit from initial value of zero to one unit of A would be = 3 – 0 =0. These profits are given in net evaluation row.

Step 3: *Improving the solution*

Column selection: Net evaluation row shows that with each unit of product A profit increase by Rs.3 and with each unit of product B profit increases by Rs.5. Basic variables does not contribute to profits at all. As objective is to maximize profits so first non basic variable to enter the solution which would improve the solution would be product B.

Row selection: Divide the maximum resource available in each row with the coefficients of selected column. Select the minimum positive value from these ratios as shown in table 4.4.1.3. For first row maximum resource was 4 but when divided by coefficient i.e. 0 it does not give any value. For second row maximum resource was 12 and when divided by 2 we get 6 hours. Lastly for third row 18 divided by 2 would give 9 hours. Thus second row is selected.

Table 4.4.1.3

Decision Variables		x_1	x_2	s_1	s_2	s_3		
Coefficients of decision variables in objective function		***3***	***5***	***0***	***0***	***0***		
Basis variables	**Coefficients of basis variables c_b in objective function**	**Coefficients of constraint functions for three plants**					**Maximum available resource**	**Ratios**
s_1	0	1	0	1	0	0	***4***	***4/0 = infinite***
s_2	0	0	2	0	1	0	***12***	***12/2 = 6 hours Selected row***
s_3	0	3	2	0	0	1	***18***	***18/2 = 9 hours***
	z_j	0	0	0	0	0		
Net evaluation row	$c_j - z_j$	3	***5 Selected column***	0	0	0		

It is important to understand that selected column represents the incoming variable whereas selected row indicates outgoing variable. So in this case s_2 would be replaced by x_2. This would be done by applying mathematical functions in such a manner that coefficients of x_2 column are replaced by coefficients of s_2 column. These are shown in following table

x_2 column coefficients	***would change into***	***s_2 column coefficients***
0		0
2	→	1
2		0

Selected row: $0x_1 + 2x_2 + 0s_1 + 1s_2 + 0s_3 = 12$

Divide all the coefficients by 2 to get: $0x_1 + 1x_2 + 0s_1 + 0.5s_2 + 0s_3 = 6$ ***(new row 2)***

To find new row 3 subtract it from twice of new row 2 i.e.:

$$(3x_1 + 2x_2 + 0s_1 + 0s_2 + 1s_3) - (0x_1 + 2x_2 + 0s_1 + 1s_2) = 18-12$$

$$3x_1 + 0x_2 + 0s_1 - 1s_2 + 1s_3 = 6 \quad \textit{(new row 3)}$$

New row 1 would remain same as coefficients are already as required i.e. 0

So, $$x_1 + 0x_2 + 1s_1 + 0s_2 + 0s_3 = 4 \quad \textit{(new row 1)}$$

Creating new simplex table with new rows would give us table 4.4.1.4. It is important to note that now x_2 has become one of the basic variables.

Table 4.4.1.4							
Decision Variables		***x_1***	***x_2***	***s_1***	***s_2***	***s_3***	
Coefficients of decision variables in objective function		***3***	***5***	***0***	***0***	***0***	
Basis variables	**Coefficients of basis variables c_b in objective function**	**Coefficients of constraint functions for three plants**					**Maximum available resource**
s_1	0	1	0	1	0	0	***4***
x_2	***5***	0	1	0	0.5	0	***6***
s_3	0	3	0	0	-1	1	***6***

Step 4: <u>*Calculating next table*</u>

For next table shown in Table 4.4.1.5 repeat the same process of finding z_j and $c_j - z_j$.

Table 4.4.1.5								
Decision Variables		***x_1***	***x_2***	***s_1***	***s_2***	***s_3***		
Coefficients of decision variables in objective function		***3***	***5***	***0***	***0***	***0***		
Basis variables	**Coefficients of basis variables c_b in objective function**	**Coefficients of constraint functions for three plants**					**Maximum available resource**	**Ratios**
s_1	0	1	0	1	0	0	***4***	***4/1 = 4***
x_2	5	0	1	0	0.5	0	***6***	***6/0 = infinite***
s_3	0	3	0	0	-1	1	***6***	***6/3 = 2 Selected row***
	z_j	0	5	0	2.5	0		
Net evaluation row	$c_j - z_j$	***3 Selected column***	0	0	-2.5	0		

This table shows x_1 to be incoming variable and s_3 as outgoing variable. So in this case s_3 would be replaced by x_1. This would be done by applying mathematical functions in such a manner that coefficients of x_1 column are replaced by coefficients of s_3 column. These are shown in following table

x_1 column coefficients	***would***	***s_3 column coefficients***

1	***change into***	0
0	$\longrightarrow$	0
3		1

Selected row: $3x_1 + 0x_2 + 0s_1 - 1s_2 + 1s_3 = 6$

New row 3 would be obtained by dividing selected row by 3 so that coefficient of x_1 i.e. 3 gets changed into 1.

So, $1x_1 + 0x_2 + 0s_1 - 0.33s_2 + 0.33s_3 = 2$ ***(new row 3)***

Subtract row 1 from new row 3 to get new row 1 i.e.

$(1x_1 + 0x_2 + 1s_1 + 0s_2 + 0s_3) - (1x_1 + 0x_2 + 0s_1 - 0.33s_2 + 0.33s_3) = 4 - 2$

$0x_1 + 0x_2 + 1s_1 + 0.33s_2 - 0.33s_3 = 2$ ***(new row 1)***

Row 2 remains same i.e. $0x_1 + 1x_2 + 0s_1 + 0.5s_2 + 0s_3 = 6$ ***(new row 2)***

Creating new simplex table with new rows would give us table 4.4.1.6. It is important to note that now x_1 has become one of the basic variables.

Table 4.4.1.6

Decision Variables		***x_1***	***x_2***	***s_1***	***s_2***	***s_3***	
Coefficients of decision variables in objective function		***3***	***5***	***0***	***0***	***0***	
Basis variables	**Coefficients of basis variables c_b in objective function**	**Coefficients of constraint functions for three plants**					**Maximum available resource**
s_1	0	0	0	1	0.33	0.33	***2***
x_2	***5***	0	1	0	0.5	0	***6***
x_1	***3***	1	0	0	-0.33	0.33	***2***

For next table shown in Table 4.4.1.7 repeat the same process of finding z_j and $c_j - z_j$.

Table 4.4.1.7

Decision Variables		***x_1***	***x_2***	***s_1***	***s_2***	***s_3***	
Coefficients of decision variables in objective function		***3***	***5***	***0***	***0***	***0***	
Basis variables	**Coefficients of basis variables c_b in objective function**	**Coefficients of constraint functions for three plants**					**Maximum available resource**
s_1	0	0	0	1	0.33	0.33	***2***
x_2	***5***	0	1	0	0.5	0	***6***
x_1	***3***	1	0	0	-0.33	0.33	***2***
	z_j	3	5	0	1.51	0.99	
Net evaluation	$c_j - z_j$	0	0	0	-1.51	-0.99	

row							

Net evaluation row of table 4.4.1.7 shows all values to be either zero or negative implying that further introduction of any variable would either does not change the solution or it would reduce the profits. ***So, this is the most optimal solution.***

The results of table 4.4.1.7 shows value of $s_1 = 2$, $x_2 = 6$ and $x_1 = 2$.

Putting these values in objective function

$$Z = 3x_1 + 5x_2 + 0s_1 + 0s_2 + 0s_3$$

We get $Z = 3*(2) + 5*(6) + 0*(2) + 0*(0) + 0*(0)$

$$Z = 6 + 30 = 36$$

Thus converting into thousands manager would earn a profit of Rs.36,000 by manufacturing 2 units of product A and 6 units of product B per week under given constraints of production capacity in three plants.

Also putting these values in following constraints we can get values of slack variables.

Plant P: $x_1 + 0x_2 + 1s_1 + 0s_2 + 0s_3 = 4$

Plant Q: $0x_1 + 2x_2 + 0s_1 + 1s_2 + 0s_3 = 12$

Plant R: $3x_1 + 2x_2 + 0s_1 + 0s_2 + 1s_3 = 18$

$s_1 = 2$ indicates plant P remains idle for 2 hours out of maximum available 4 hours.

$s_2 = 0$ indicates maximum available resources of plant Q are fully utilized.

$s_3 = 0$ indicates maximum available resources of plant R are fully utilized.

Exercise 2

1. The $c_j - z_j$ of a simplex tableau gives

a. The number of units of each basic variable that must be removed from the solution if a new variable is entered

b. The gross profit/loss given up by adding one unit of a variable into the solution

c. The next profit or loss that will result from introducing one unit of the variable indicated in that column into the solution

2. Which of the following is NOT true about slack variables in a simplex tableau?

a. They are used to convert less than and equal to constraint inequalities to equations

b. They represent unused resources

c. They require the addition of an artificial variable

d. They yield no profit

3. Which of the following could not be a linear programming problem constraint?

a. 1A + 2B < 5   b. 1A + 2B <= 3   c. 1A + 2B = 3

d. 1A + 2B + 3C + 4D<= 5   e. 1A + 3B >= 6

2.4.2 Graphical Method:

The same illustration is solved by using Graphical Method. The formulated model was:

Maximize	$Z = 3x_1 + 5x_2$			
Subject to:	***Plant P:***	x_1		$<= 4$
	Plant Q:	$2x_2$		$<= 12$
	Plant R:	$3x_1$ +	$2x_2$	$<= 18$
and	$x_1, x_2 >= 0$			

It is important to note that inequalities have to be converted into equalities to solve graphically.

Plant P:	x_1		$= 4$
Plant Q:	$2x_2$		$= 12$
Plant R:	$3x_1$ +	$2x_2$	$= 18$

Following steps are followed:

<u>Step 1:</u> For first equation of plant P put x_1 as 0 and find x_2 which would be in this case to be 0. Then put x_2 as 0 and we get x_1 as 4.

So points for line 1 are: (0,0) and (4,0)

For second equation of plant Q when x_1 is being put as 0 x_2 was 6 and when x_2 was put as 0 x_1 was 0.

So points for line 2 are: (0,6) and (0,0)

For third equation of plant R when x_1 was put as 0 x_2 was 9 and when x_2 was put to 0 x_1 was 6.

So points for line 3 are: (0,9) and (6,0)

These three lines are drawn on the graph. X axis of graph is marked as x_1 indicating units of product A and Y axis of graph is marked as x_2 indicating units of product B.

<u>Step 2:</u> The intersection points of these three lines are identified which would give values of x_1 and x_2. As in the original model all constraints had less than or equal to inequality so for all lines area under the line i.e. towards the origin would be identified. The common area of all three lines would be identified and intersecting points would be marked. The common area is shown as darkened block in table 4.4.2.1

<u>Step 3:</u> As objective is of maximization look for highest intersecting point in the common area. In this case such highest point is (2,6).

<u>Step 4:</u> Put these values in the objective function Z

$$Z = 3x_1 + 5x_2$$

$$Z = 3*(2) + 5*(6) = 36$$

Feasibility can be checked by putting these values in constraint functions i.e.

Plant P:	x_1		$<= 4$	
	2	<	4	*(fulfilled)*
Plant Q:	$2x_2$		$<= 12$	

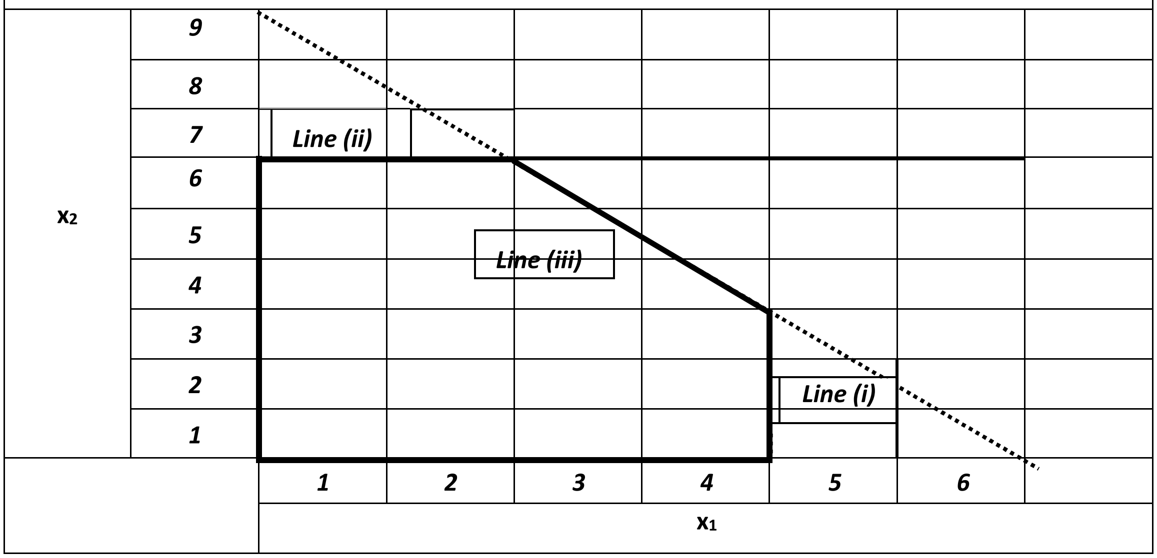
$2*6 = 12 = 12$ *(fulfilled)*

Plant R: $3x_1 + 2x_2 \quad <= 18$

$3*2 + 2*6 = 18 = 18$ *(fulfilled)*

As objective is to maximize Z so point 3 indicating 2 units of product A and 6 units of product B would be manufactured to get maximum profit of Rs.36,000.

Table 4.4.2.1



2.5 <u>PRODUCT MIX LINEAR PROGRAMMING PROBLEM: AN ILLUSTRATION OF MINIMIZATION OBJECTIVE</u>

Chemical company is producing two kinds of chemicals A and B used in manufacturing of soaps and detergents. The company estimated that in coming month combined demand for products A and B would be at least 350 gallons. Also, company already has a order of at least 125 gallons of product A from a major customer. So minimum of 125 gallons of A has to be manufactured. Maximum of 600 hours of production time is available next month. It has also been identified that one unit of product A requires 2 hours of production whereas one unit of B would be manufactured in 1 hour. Also, production cost for one unit of A was estimated to be Rs.2 and that of B was Rs.3. Company wanted to formulate a production schedule in such a manner so that production cost is minimized in production of certain units of A and B. thus, operation research team was given the task of finding out maximum number of units of A and B to be manufactured under given constraints so that cost is minimized.

This illustration is solved both by Simplex and Graphical method.

2.5.1 Simplex Method: Simplex method requires following steps for solving of linear programming problem

<u>Step 1</u>: *<u>Formulation of linear programming model</u>*

x_1 = number of units of product A produced per week

x_2 = number of units of product B produced per week

Z = total cost per month from producing A and B

Minimize $Z = 2x_1 + 3x_2$

Subject to:

- Customers' minimum demand: $x_1 \quad >= 125$
- Minimum combined production of both products for coming month:

  $1x_1 + 1x_2 \quad >= 350$
- Maximum available production time for next month:

  $2x_1 + 1x_2 \quad <= 600$

and $x_1, x_2 >= 0$

An important aspect to be noticed in these constraint equations is that inequality is of greater than. This implies that left hand side could be greater than or equal than right hand side. Thus, for example customers' minimum demand is 125 gallons and it could be more. So, to make equations equal a variable has to be subtracted which is termed as ***surplus variable.*** For inequalities less than or equal to slack variables are added.

This results in following linear programming model.

Minimize $Z = 2x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3$

Subject to:

$$x_1 + 0x_2 - 1s_1 + 0s_2 + 0s_3 = 125$$
$$1x_1 + 1x_2 + 0s_1 - 1s_2 + 0s_3 = 350$$
$$2x_1 + 1x_2 + 0s_1 + 0s_2 + 1s_3 = 600$$

To check feasibility of solution x_1 and x_2 are put as zero. This would give values of surplus variables i.e. s_1 and s_2 and of slack variables s_3 as:

$s_1 = -125$; $s_2 = -350$ and $s_3 = 600$

Negative surplus variables make the solution infeasible as it does not fulfill non-negativity criteria. To do so artificial variables a_1 and a_2 in addition to surplus variables are added in the equation. Value of these variables is bigger than surplus variables so that they become positive. Thus objective and constraint functions are formulated as:

Minimize $Z = 2x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3 + Ma_1 + Ma_2$

Subject to:

$$x_1 + 0x_2 - 1s_1 + 0s_2 + 0s_3 + 1a_1 + 0a_2 = 125$$
$$1x_1 + 1x_2 + 0s_1 - 1s_2 + 0s_3 + 0a_1 + 1a_2 = 350$$
$$2x_1 + 1x_2 + 0s_1 + 0s_2 + 1s_3 + 0a_1 + 0a_2 = 600$$

Now we have three constraint equations to solve for seven variables namely, decision variables (x_1 and x_2), surplus variables (s_1 and s_2), slack variable (s_3) and artificial variables (a_1 and a_2).

This is done by assuming $x_1 = x_2 = s_1 = s_2 = 0$

This would give us ***basis variables***: $a_1 = 125$; $a_2 = 350$ and $s_3 = 600$

Thus all variables have become positive or zero making solution to be feasible. Also it gives initial basic feasible solution which indicates the beginning of operations when production of two products under consideration is assumed to be zero.

The formulated can be expressed in simplex table form as shown in table 4.5.1.1

Table 4.5.1.1

Decision Variables		x_1	x_2	s_1	s_2	s_3	a_1	a_2	
Coefficients of decision variables in objective function		***2***	***3***	***0***	***0***	***0***	M	M	
Basis variables	**Coefficients of basis variables c_b in objective function**	**Coefficients of constraint functions for three plants**							**Maximum available resource**
a_1	M	1	0	-1	0	0	1	0	***125***
a_2	M	1	1	0	-1	0	0	1	***350***
s_3	0	2	1	0	0	1	0	0	***600***

Same procedure is adopted to find an optimal solution as discussed in section 4.4.1. as the problem is of minimization here we would show only simplex table for first solution.

First solution: Table 4.5.1.2 shows calculations of z_j and c_j-z_j i.e. net evaluation row.

Table 4.5.1.2

Decision Variables		x_1	x_2	s_1	s_2	s_3	a_1	a_2		
Coefficients of decision variables in objective function		***2***	***3***	***0***	***0***	***0***	M	M		
Basis variables	**Coefficients of basis variables c_b in objective function**	**Coefficients of constraint functions for three plants**							**Maximum available resource**	**Ratio**
a_1	M	1	0	-1	0	0	1	0	***125***	***125/1 = 125***
a_2	M	1	1	0	-1	0	0	1	***350***	***350/1 = 350***
s_3	0	2	1	0	0	1	0	0	***600***	***600/2 = 300***
	z_j	2M	M	-M	-M	0	M	M		
Net evaluation row	c_j-z_j	2-2M	3-M	M	M	0	0	0		

As the objective is of minimization so in net evaluation row select maximum negative value. In this case if we assume M to have very high value like 10,000 then column 1 would give us least value. So in first solution column 1 is selected. It should be noted that in case of maximization LPP net evaluation row with maximum positive value is selected.

M is assumed to carry very big values so that artificial variable associated with it goes out of the solution as early as possible. This is done because artificial variable is an imaginary variable used for the purpose of making solution feasible. It does not contribute in any way to the objective function. ***That is why this method is also called as Big M method.*** *Another point to be noted is that artificial variable is added in the objective function if function is of minimization but it is subtracted if objective function is of maximization.*

So first column with least value and first row with minimum resource is selected. This would make artificial variable a_1 as the outgoing variable and x_1 as the incoming variable.

Solving mathematically and introducing new variables to replace basis variables final solution can be obtained. ***The solution reaches optimal stage when all values in net evaluation row become positive.***

Second solution: Table 4.5.13 shows next iteration. It is important to note that row to be selected depends on least positive value. So, a_2 would be replaced by x_2.

Table 4.5.1.3									
Decision Variables		x_1	x_2	s_1	s_2	s_3	a_2		
Coefficients of decision variables in objective function		***2***	***3***	***0***	***0***	***0***	M		
Basis variables	**Coefficients of basis variables c_b in objective function**	**Coefficients of constraint functions for three plants**						**Maximum available resource**	**Ratio**
x_1	**2**	1	0	-1	0	0	0	***125***	***undefined***
a_2	M	0	1	0	0	0	1	***225***	***225***
s_3	0	0	1	2	0	1	0	***350***	***-350***
	z_j	2	M	-2	0	0	M		
Net evaluation row	c_j-z_j	0	3-M	2	0	0	0		

Third solution:

Table 4.5.1.4						
Decision Variables	x_1	x_2	s_1	s_2	s_3	
Coefficients of decision variables in objective function	***2***	***3***	***0***	***0***	***0***	

Basis variables	Coefficients of basis variables c_b in objective function	Coefficients of constraint functions for three plants					Maximum available resource
x_1	***2***	1	0	-1	0	0	***125***
x_2	***3***	0	1	0	0	0	***225***
s_3	0	0	0	2	0	1	***125***
	z_j	2	3	-2	0	0	
Net evaluation row	c_j-z_j	0	0	2	0	0	

As all the values in net evaluation row are either positive or zero so cost cannot be reduced further. Thus minimum cost can be achieved by manufacturing **125 units of A and 225 units of B**.

So Minimum Cost Z $= 2x_1 + 3x_2$

Z $= 2*(125) +3*(225)$

$=$ Rs. 925

Putting values of $x_1 = 125$, $x_2 = 225$ and $s_3 = 125$ in constraint functions we can check the feasibility of solution:

$x_1 + 0x_2 - 1s_1 + 0s_2 + 0s_3 = 125$

Will give s1 = 0 implying major customers' minimum demand is fulfilled.

$1x_1 + 1x_2 + 0s_1 - 1s_2 + 0s_3 = 350$

Will give s2 = 0 implying that minimum combined production is fulfilled.

$2x_1 + 1x_2 + 0s_1 + 0s_2 + 1s_3 = 600$

$600 = 600$

Thus, satisfying all the constraints.

2.5.2 Graphical Method: The above formulated model has been illustrated below:

Minimize $Z = 2x_1 + 3x_2$

Subject to:

- Customers' minimum demand:

  $x_1 >= 125$

- Minimum combined production of both products for coming month:

  $1x_1 + 1x_2 >= 350$

- Maximum available production time for next month:

  $2x_1 + 1x_2 <= 600$

and $x_1, x_2 >= 0$

inequalities have been converted into equalities:

- ***Line (i)*** $x_1 + 0x_2 = 125$
- ***Line (ii)*** $1x_1 + 1x_2 = 350$
- ***Line (iii)*** $2x_1 + 1x_2 = 600$

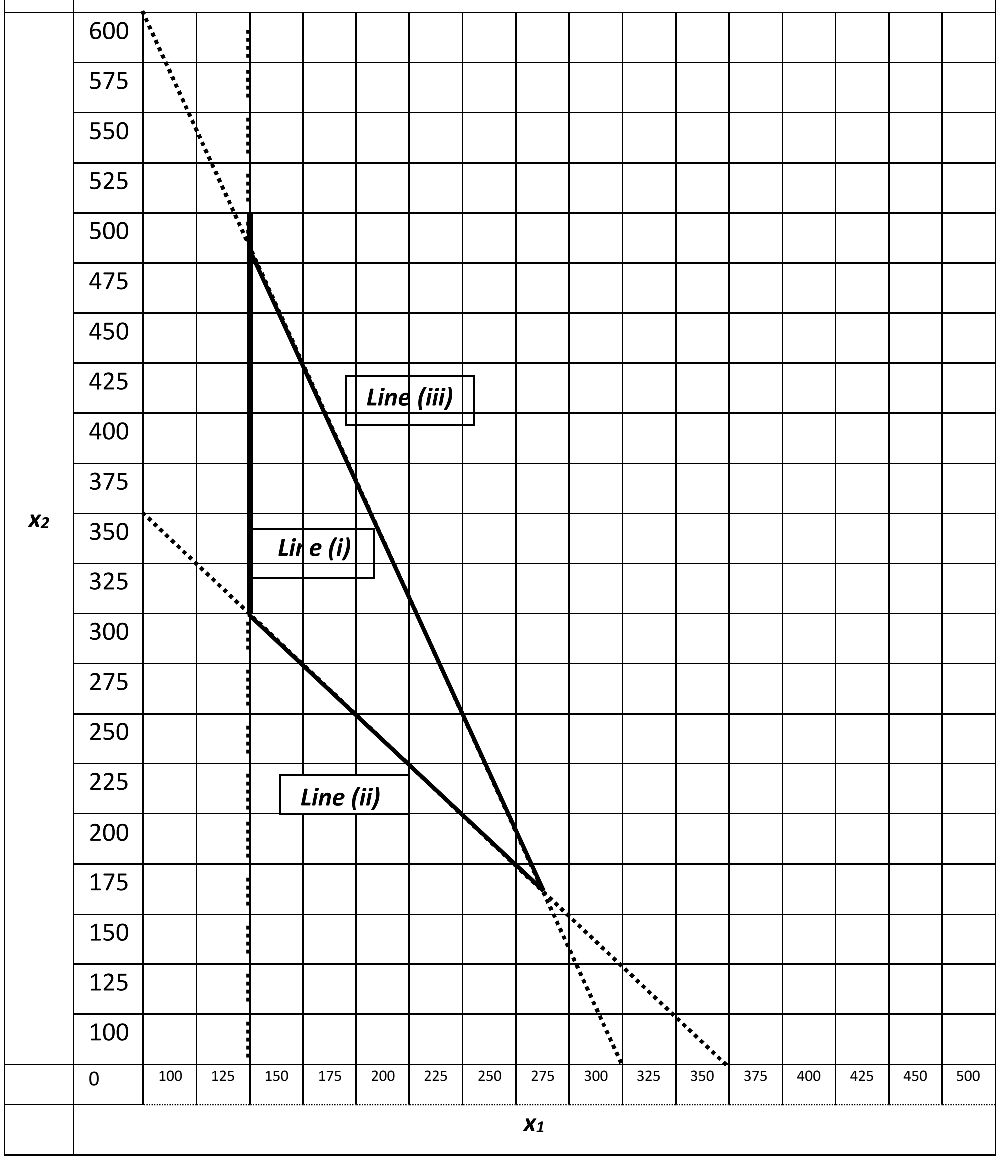
For (i) points required to draw line of equation would be (0,0) and (125,0)

For (ii) points required to draw line of equation would be (0,350) and (350,0)

For (iii) points required to draw line of equation would be (0,600) and (300,0)

These lines should be drawn on a graph paper. These lines have been shown in table 4.5.2.1.

Table 4.5.2.1



Graphically the objective is to find the common area of three constraints depicted by lines drawn on the graph paper. Line (i) indicates an inequality which is greater than representing minimum demand of a major customer. Thus, area represented by it would be entire area starting from line away from the origin. Line (ii) again represents a greater than or equal to inequality. So its area would also be starting from the line away from origin. These two areas have infinite limit as there are not bounded by any upper limit. Line (iii) represents a less than or equal to inequality. So its area would be starting from line towards the origin. This has a bounded area originating from start of line and ending at the origin. Common area represented by three lines is shown in darkened block.

Now objective is to minimize cost so Z would be calculated at all intersecting points and point giving minimum value of Z would indicate number of units to be produced for A and B.

First point is x_1 = 125 and x_2 = 300

Feasibility of x_1 and x_2 can be checked by putting these values in given constraints i.e.

(i) x_1 >= 125

125 = 125 *(fulfilled)*

(ii) $1x_1 + 1x_2$ >= 350

1*125 + 1*300 = 425 > 350 *(fulfilled)*

(iii) $2x_1 + 1x_2$ <= 600

2*125 + 1* 300 = 550 < 600 *(fulfilled)*

As all the conditions are fulfilled so feasible solution would be:

Z = 2*125 + 3*300 = 1150 and number of units of A and B would be 125 and 300 respectively

Second point is x_1 = 125 and x_2 = 475 and

(i) x_1 >= 125

125 = 125 *(fulfilled)*

(ii) $1x_1 + 1x_2$ >= 350

1*125 + 1*475 = 600 > 350 *(fulfilled)*

(iii) $2x_1 + 1x_2$ <= 600

2*125 + 1* 475 = 725 is not less than or equal to 600. *Thus this solution is not feasible.*

Third point is x_1 = 260 (approx.) and x_2 = 160 (approx.)

(i) x_1 >= 125

260 > 125 *(fulfilled)*

(ii) $1x_1 + 1x_2$ >= 350

1*260 + 1*160 = 420 > 350 *(fulfilled)*

(iii) $2x_1 + 1x_2$ <= 600

2*260 + 1* 160 = 680 is not less than or equal to 600. *Thus this solution is not feasible.*

Thus objective of minimizing cost is achieved by manufacturing 125 units of A and 300 units of B.

Exercise 3

1. Considering the following linear programming problem:

Maximize: $40 X1+ 30 X2 + 60X3$

Subject to: $X1 + X2 + X3 >=90$, $12X1 + 8X2 + 10X3 <= 1500$, $X1, X2, X3>0$

How many slack, surplus, and artificial variables would be necessary if the simplex were used to solve the problem?

a. 3 slack, 3 surplus, and 3 artificial
b. 1 slack, 2 surplus, and 2 artificial
c. 1 slack, 4 surplus, and 4 artificial
d. 1 slack, 1 surplus, and 1 artificial

2. Let xij = gallons of component i used in gasoline j. Assume that we have two components and two types of gasoline. There are 8,000 gallons of component 1 available, and the demand gasoline types 1 and 2 are 11,000 and 14,000 gallons respectively. Write the constraint stating that the component 1 cannot account for more than 35% of the gasoline type 1.

a. $x11 + x12<= (0.35)(x11 + x21)$
b. $x11>= 0.35 (x11 + x21)$
c. $x11<= 0.35 (x11 + x12)$
d. $-0.65x11 + 0.35x21 <=0$
e. $0.65x11 - 0.35x21<= 0$

2.6 SOME ILLUSTRATIONS OF FORMULATING LINEAR PROGRAMMING PROBLEM

Example 1:

An individual decided to go on steady diet of two foods (A and B). He realizes that this isn't the healthiest diet, so he wants to make sure that he eats the right quantities of the two foods to satisfy some key nutritional requirements. He has obtained the following nutritional and cost information:

	Grams of Ingredient per serving		
Ingredient	**A**	**B**	**Daily**
Carbohydrates	5	15	>=50
Protein	20	5	>=60
Fat	15	2	<=60
Cost/serving	Rs.4	Rs.2	

Formulate a linear programming model to determine the number of daily servings of A and B that will meet these requirements at a minimum cost.

Solution:

Minimize $Z = 4A + 2B$

Subject to:
$5A + 15B >=50$
$20A + 5B >=60$
$15A + 2B <=60$

Model formulation:

$$5A + 15B -1.s_1 + 0.s_2 + 0.s_3 + 1.a_1 + 0.a_2 = 50$$

$20A + 5B + 0.s_1 - 1.s_2 + 0.s_3 + 0.a_1 + 1.a_2 = 60$

$15A + 2B + 0.s_1 + 0.s_2 + 1.s_3 + 0.a_1 + 0.a_2 = 60$

And $Z = 4A + 2B + 0.s_1 + 0.s_2 + 0.s_3 + Ma_1 + Ma_2$

Example 2:

Maximize $Z = 50A + 40B$

Subject to: $3A + 5B \leq 150$

$0A + 1B \leq 20$

$8A + 5B \leq 300$

$1A + 1B \geq 25$

Model formulation:

$3A + 5B + 1.s_1 + 0.s_2 + 0.s_3 + 0.s_4 + 0.a_4 = 150$

$0A + 1B + 0.s_1 + 1.s_2 + 0.s_3 + 0.s_4 + 0.a_4 = 20$

$8A + 5B + 0.s_1 + 0.s_2 + 1.s_3 + 0.s_4 + 0.a_4 = 300$

$1A + 1B + 0.s_1 + 0.s_2 + 1.s_3 - 1.s_4 + 1.a_4 = 25$

And $Z = 50A + 40B + 0.s_1 + 0.s_2 + 0.s_3 + 0.s_4 – Ma_4$

2.7 SUMMARY

Linear Programming Problem (LPP) is an effective method of solving number of operations problem. This chapter primarily focuses on product mix problem where limited resources need to be allocated for production of more than one product. LPP should fulfill various assumptions of additivity, proportionality, certainty and divisibility. These problems can be used to fulfill the objectives of either minimization of cost or maximization of profits. Such objectives depend on decision variables and constraint functions. Proper formulation of model is most important step in linear programming model as it decides on addition of slack or surplus variables. The model can be solved both y simplex and graphical method. Graphical method has a limitation of its application to only two decision variables. Thus, if a LPP has more than two decision variables then Simplex method is more accurate way of finding solution. This chapter has illustrated both methods by taking example of minimization of cost and maximization of profits.

2.8 GLOSSARY

- ***Linear programming model:*** is a model with linear objective function, a set of linear constraints and non-negative variables.
- ***Constraints:*** are set of equality or inequality equations indicating minimum or maximum availability of resources.

- ***Decision variable:*** are the variables which contribute to the objective function. These are controllable inputs decided by set of constraints under which objective has to be fulfilled.
- ***Slack variable:*** indicates amount of unused or idle resource. It is represented by adding a variable to the left side of constraint function with less than or equal to inequality.
- ***Surplus variable:*** indicates amount of resource available over and above certain required level. It is represented by subtracting a variable to the left side of constraint function with greater than or equal to inequality.

2.9 ANSWERS TO CHECK YOUR PROGRESS/ SELF ASSESSMENT EXERCISE

Exercise 1

1. D
2. D
3. C

Exercise 2

1. C
2. C
3. A

Exercise 3

1. D
2. C

2.10 REFERENCES/ SUGGESTED READINGS

- Hillier, F.S. and Lieberman, J. G., *Operations Research,* Tata McGraw Hill, 2009, 10th Reprint.
- Anderson, D.R., Sweeney D. J. and Williams, T.A., *An Introduction to Management Science: Quantitative Approach to Decision Making,* South Western Cengage Learning, 11th Edition.
- Natarajan, A.M., Balasubramani, P. and Tamilarasi, A., *Operations Research,* Pearson, 2012, 9th Edition.

2.11 TERMINAL AND MODEL QUESTIONS

1. The production manager for a soft drink company is considering the production of 2 kinds of soft drinks: regular and diet. Two of her limited resources are production time (8 hours = 480 minutes per day) and syrup (1 of her ingredients) limited to 675 gallons per day. To produce a regular case requires 2 minutes and 5 gallons of syrup, while a diet case needs 4 minutes and 3 gallons of syrup. Profits for regular soft drink are Rs.3.00 per case and profits for diet soft drink are Rs.2.00 per case. Determine number of two types of soft drinks to be produced so as to achieve maximum profit.

2. A factory makes 3 components, A, B and C using the same production process for each. A unit of A take 1 hr, a unit of B takes 0.75 hrs and a unit of C takes 0.5 hrs. In addition, C has to be hand finished, an activity taking 0.25 hrs per unit. Each week total production time (excluding hand finishing) must not exceed 300 hrs and hand finishing must not exceed 45 hrs. The components are finally assembled to make two finished products. One product consists of 1 unit of A and 1 unit of C selling for 30 pounds whilst the other consists of 2 units of B and 1 unit of C and sells for 45 pounds. At most 130 of the first product and 100 of the second product can be sold each week. Formulate the problem of planning weekly production to maximize total proceeds as a linear programming problem in 2 variables and obtain the solution graphically.
3. A coffee packer blends Brazilian coffee and Columbian coffee to prepare two products, super and deluxe brands. Each kilogram of super coffee contains 0.5 kg of Brazilian coffee and 0.5 kg of Colombian coffee, whereas each kilogram of deluxe coffee contains 0.25 kg of Brazilian coffee and 0.75 kg of Colombian coffee. The packer has 120 kg of Brazilian coffee and 160 kg of Colombian coffee on hand. If the profit one kg of super coffee is 22 cents and the profit on each kg of Deluxe coffee is 30 cents, how many kg of each type of coffee should be blended to maximize profits?

Chapter 3: TRANSPORTATION MODEL

3.0 OBJECTIVES

This chapter should help students to understand:

- Functions of transportation problem
- Assumptions of transportation problem
- Various models of transportation problem
- Methods of finding an initial solution and checking its feasibility.

3.1 INTRODUCTION

Transportation problem is a special kind of linear programming problem which is used to determine number of units to be distributed from limited supply centers to known demand centers. In majority of cases there are more than one supply and demand centers. This makes problem of allocating number of units from one supply center to a particular demand center peculiar. With increase in number of supply and demand centers the problem becomes more complex. Typically the objective of a transportation model is to find minimum cost required to fulfill demand of various destinations when material is sourced from various suppliers. For example, a state having multiple thermal plants can source coal from various coal mines. Requirement of coal of each thermal plant is known. The question to be answered is how much coal needs to be sourced from which supply center so that total transportation cost comes out to be minimum. An FMCG company like Hindustan Lever manufacturing surf sources raw material from number of suppliers and distributes final product to number of retailers. In such cases transportation model finds its application to determine number of units to be sourced from one supplier and to be supplied to a particular distributor at minimum or most optimal transportation cost.

3.2 PARAMETERS AND STRUCTURE OF TRANSPORTATION MODEL

A transportation model specifically includes following parameters:

- Limited number of suppliers and capacity of each supplier.
- Known number of demand centers with known demand of each center.
- Known cost of transporting one unit from each supply to each demand center.

These three elements are shown in the Table 6.2.1 indicating generalized structure of a transportation model.

Table 6.2.1

	Demand centers				
Supply centers	***D1***	***D2***	***D3***	***D4***	***Total Supply of each center***
S1	a_{11}	a_{12}	a_{13}	a_{14}	x_1
S2	a_{21}	a_{22}	a_{23}	a_{24}	x_2
S3	a_{31}	a_{32}	a_{33}	a_{34}	x_3
Total Demand of each center	y_1	y_2	y_3	y_4	

Table 6.2.1 illustrates:

- Four demand centers: D1, D2, D3 and D4.
- Three supply centers: S1, S2 and S3.
- Total supply of each supply center: Supply of S1 = x_1, Supply of S2 = x_2 and Supply of S3 = x_3
- Total demand of each center: Demand of D1 = y_1, demand of D2 = y_2 and demand of D3 = y_3
- Cost of transporting one unit from each supply to each demand center. a_{11} is cost of transporting one unit from S1 to D1 and so forth.

This structure would be used for solving of all transportation problems in this chapter.

3.3 ASSUMPTIONS OF TRANSPORTATION PROBLEMS

Requirement assumption: Each supplier has a fixed number of units to be supplied i.e. it has a fixed output which has to be supplied to various destinations. Each demand center has a fixed demand which is to be met from number of units supplied from various and limited suppliers.

Feasibility: A transportation problem is considered to be feasible if total output from various supply sources equals total demand of various destinations. From table 6.2.1 this can be illustrated as:

$$x_1 + x_2 + x_3 = y_1 + y_2 + y_3 + y_4$$

In some cases this assumption gets violated as practically each source indicates its maximum capacity to produce and each demand center indicates its maximum demand. However, such a situation can be resolved which would be discussed later in detail.

Cost assumption: the cost of transporting units from one source to a particular destination is directly proportional to number of units to be transported. For example, if transportation cost per unit from one source to a destination is Rs.5 then transporting ten units would be Rs.50.

Integer values: A transportation problem would have every demand and supply value as integer value. Number of units allocated to each destination and cost of transporting are also integer values.

3.4 VARIATIONS OF TRANSPORTATION MODEL

Unbalanced problem: Such a problem occurs when total units to be transported i.e. supply is not equal to total units required i.e. demand. If total supply is more than total demand then excess supply would be shown as slack in the model. But problem cannot be resolved if total supply is less than total demand. In such cases model would have an infeasible solution. This is resolved by adding a dummy supplier with supply equal to difference between total demand and total supply. The costs associated with such supplier are assumed to be zero. So, transportation cost of each unit from such supplier to various destinations would be zero so that total transportation cost depicts cost of transporting actual number of units. Such a supplier is imaginary and in reality does not exist.

Maximization objective function: In some transportation problems the objective is find a solution to maximize profits or revenues. In such cases cost cells are converted to profit or revenue cells. The problem is solved in similar fashion as it does not affect the assumptions of basic transportation model.

Degenerate problem: Degeneracy occurs in a transportation problem when number of cells allocated with certain number of units is not equal to m + n -1 where 'm' is number of rows and 'n' is number of columns. Such a problem is resolved by allocating least number of units to minimum cost cell.

These variations would be discussed with illustrations in following sections.

Exercise 1

1. The product to be transported from multiple sources to multiple destinations should be same.

   (a) True (b) False

2. Transportation problem can be used to fulfill which of the following objectives:

   (a) Minimization of cost (b) Maximization of profits (c) Both

3. The capacity and requirement of supply and demand centers should be

   (a) Known (b) limited (c) Both

3.5 STEPS IN TRANSPORTATION MODEL

Step 1: *Formulate a transportation model:* In first step check the assumptions of transportation model. All assumptions namely requirement of known and limited number of suppliers and destinations, total demand is equal to total supply, known cost of transporting one unit from each supply to each demand center and integer values of all coefficients should be met.

Step 2: *Find Initial Basic Feasible solution (IBFS):* After fulfilling assumptions an initial solution to transportation problem is found by using any of the following methods. The objective here is to achieve objective of minimum cost or maximum profits.

- Least Cost Method
- North-West Corner method
- Vogel's approximation method (VAM)

Step 3: *Perform Optimality test:* Is the solution achieved in second step is the best solution? Can another solution be found which reduces cost or increases profit further? To find the best or most optimal solution Modified Distribution (MODI) method is used to check optimality of found solution.

Step 4: *Iteration:* Step 3 is repeated till final and most optimal solution is achieved.

3.6 BASIC TRANSPORTATION PROBLEM

Following is the illustration of transportation problem when all the conditions or assumptions are met.

Table 6.6.1

	Demand centers				
Supply centers	***D1***	***D2***	***D3***	***D4***	***Total Supply of each center***
S1	3	2	7	6	5000
S2	7	5	2	3	6000
S3	2	5	4	5	2500
Total Demand of each center	6000	4000	2000	1500	

Table 6.6.1 shows transportation cost of each unit to be distributed from three supply centers S1, S2 and S3 to four demand centers D1, D2, D3 and D4. The capacity or output of each supply center is 5000, 6000 and 2500 units respectively. Similarly, demand or requirement of each demand center is 6000, 4000, 2000 and 1500 units respectively. Objective is to find most optimum transportation cost.

Step 1: *Formulate a transportation model:*

Total demand = 6000 + 4000 + 2000 + 1500 = 13,500

Total supply = 5000 + 6000 + 2500 = 13,500

As total demand is equal to total supply and also other assumptions are fulfilled so we can find initial basic feasible solution.

Step 2: *Find Initial Basic Feasible solution (IBFS):*

3.6.1 *Least Cost Method:* This method involves following steps:

- Find the least cost cell in entire cost matrix.
- Allocate number of units required or available. If demand is more than supply assign number of available units and if supply is more than demand assign number of units required.
- In case of tie of minimum cost cell select the cell where maximum units can be assigned as intention would be to transport maximum number of units at least cost.

In above problem (Table 6.6.1) least cost cell is of Rs.2. But there are three such cells indicated by path S1---D2, S2---D3 and S3---D1. So which cell should be selected?

For path S1---D2 requirement is of 4000 units and available are 5000 units. As supply is more than demand so its demand of 4000 can be fulfilled from D2.

For path S2---D3 demand of 2000 units can be fulfilled from D3. As supply is more than demand so 2000 units can be assigned to respective cell.

For path S3---D1 available units of 6000 is more than required demand of 2500 units. As demand is more than supply so only 2500 are assigned.

As discussed, in case of tie select the cell where maximum units can be allocated. In this case 4000 is the maximum number of units that can be allocated.

(i) 4000 units are assigned to cell a_{12}. Now demand of D2 is fulfilled so it would not be considered in next allocation. Its requirement becomes zero from 4000 and supply of S1 reduces from 5000 to 1000 as 4000 units have been allocated to D2.

(ii) After striking off column 2 next cost to be assigned was found to be cell a_{31}. It has been assigned 2500 units. This would exhaust supply of S3 and D1 would be left with 3500 units out of 6000. This shows that demand of D1 is still not completely fulfilled. Remaining demand would be fulfilled by other supply centers.

(iii) Now cell a_{23} would be assigned 2000 units as its demand is 2000 and supply is 6000. So column 3 would be struck off and supply of S2 would reduce from 6000 to 4000.

(iv) Next least cost cell is of Rs.3. There is a tie again as cell a_{11} and cell a_{24} have cost of Rs.3. By applying rule as discussed cell a_{23} is assigned 1500 units. Demand of D4 centre is exhausted so column 4 would be struck off and supply of S2 would reduce to 4500 from 6000 units.

(v) Cell a_{11} would be assigned 1000 units. Remaining demand of D1 was 3500 units but only 1000 units from S1 were available. As S1 gets exhausted so it would be struck off and demand of D1 would reduce to 2500 from 3500.

(vi) Now only cell a_{21} is left unallocated. Remaining demand of D1 of 2500 units would be fulfilled from remaining supply of S2 i.e. 2500 units.

By following above mentioned procedure we get following table 6.6.1.1. Dashed lines indicated columns or rows that have been struck off after their demand or supply gets exhausted.

Table 3.6.1.1					
	Demand centers				
Supply centers	***D1***	***D2***	***D3***	***D4***	***Total Supply of each center***
S1	3 (1000)	2 (4000)	7	6	***5000 / (1000) / (0) (v)***
S2	7 (2500)	5	2 (2000)	3 (1500)	***6000 / (4000) / (2500) / (0) (vi)***
S3	2 (2500)	5	4	5	***2500 / (0) (ii)***
Total Demand of each center	***6000 / (3500) / (2500) / (0)***	***4000 / (0) (i)***	***2000 / (0) (iii)***	***1500 / (0) (iv)***	

Total transportation cost = 3 * 1000 + 2*4000 + 7*2500 + 2*2000 + 2*2500 + 3*1500

= 3000 + 8000 + 17500 + 4000 + 5000 + 4500

= Rs.42,000

3.6.2 ***<u>North-West Corner Method:</u>*** This method involves following steps:

- Select the north-west cell i.e. upper left corner cost cell marked a_{11}.
- Allocate number of units required or available. If demand is more than supply assign number of available units and if supply is more than demand assign number of units required.
- If demand is more than supply set the cell equal to supply and proceed *<u>vertically</u>*.
- If demand is less than supply set the cell equal to demand and proceed *<u>horizontally</u>*.
- If demand is equal to supply proceed *<u>diagonally</u>*.

Illustration for north-west corner method is provided on same example shown in table 6.6.1

(i) In cell a_{11} 5000 units are assigned. Supply from S1 gets fully exhausted so S1 centre is struck off. As demand is more than supply next allocation would be done by proceeding vertically.

(ii) Next cell is a_{21}. Assign remaining 1000 units to the cell. Demand from D1 gets fully exhausted so D1 centre is struck off. As supply is more than demand so proceed horizontally.

(iii) Next cell is a_{22}. Assign 4000 units to cell as its demand is 4000 and available is 5000. Demand from D2 gets fully exhausted so D2 centre is struck off. As demand is less than supply so proceed horizontally.

(iv) Next cell is a_{23}. 1000 units are assigned to this cell. Supply from S2 gets fully exhausted so S2 centre is struck off. As demand is more than supply so proceed vertically.

(v) Next cell is a_{33}. 1000 units are assigned to this cell. Demand from D3 gets fully exhausted so D3 centre is struck off. As demand is less than supply so proceed horizontally.

(vi) Next cell is a_{34}. Assign 1500 units to this cell.

By following above mentioned procedure we get following table 5.6.2.1. Dashed lines indicated columns or rows that have been struck off after their demand or supply gets exhausted.

Table 6.6.2.1

	Demand centers				
Supply centers	***D1***	***D2***	***D3***	***D4***	***Total Supply of each center***
S1	3 (5000)	2	7	6	***5000 / (0) (i)***
S2	7 (1000)	5 (4000)	2 (1000)	3	***6000 / (5000) / (1000) / (0) (iv)***
S3	2	5	4 (1000)	5 (1500)	***2500 / (1500) / (0)***
Total Demand of each center	***6000 / (1000) / (0) (ii)***	***4000 / (0) (iii)***	***2000 / (1000) / (0) (v)***	***1500 / (0) (vi)***	

Total transportation cost = 3*5000 + 7*1000 + 5*4000 + 2*1000 + 4*1000 + 5*1500

= 15000 + 7000 + 20000 + 2000 + 4000 + 7500

= Rs.55,500.

3.6.3 ***<u>Vogel's approximation method (VAM):</u>*** This method involves following steps:

- Find the difference between smallest and second smallest cost cell for each row and column.

- Select the row or column with greatest difference.
- In the selected row or column assign number of units to the least cost cell. If demand is more than supply assign number of available units and if supply is more than demand assign number of units required.
- If there is a tie in the difference between smallest and second smallest cost cell then select row or column with minimum cost cell. If there is further tie between minimum cost cell then select that row or column where maximum units can be allocated to that tied cell.

Illustration for VAM is provided on same example shown in table 6.6.1

(i) r1 indicates difference between smallest and second smallest cost cell of each row. c1 indicates similar difference from each column. Maximum difference comes out for second column i.e. of D2. In that column minimum cost cell of Rs.2 is in the cell a_{12}. Its demand is of 4000 units and supply is 5000, so 4000 units are supplied from S1 to D2. As entire demand gets fulfilled so D2 is struck off.

(ii) Differences between cost cells are again found in similar fashion indicated by r2 and c2. Maximum difference of 3 comes out for S1. In this row minimum cost cell of Rs.3 is a_{11}. Remaining 1000 units are allocated to this cell.

(iii) Next differences are indicated by r3 and c3. Maximum difference of 5 occurs for D1. Minimum cost cell of Rs.2 represented by a_{31} is allocated 2500 units exhausting supply of S3.

(iv) Only cost cells of S2 i.e. only one row is left unallocated. By applying least cost method first allocation is done to cell a23 of 2000 units, then to cell a24 of 1500 units and lastly to cell a21 of 2500 units.

By following above mentioned procedure we get following table 6.6.3.1. Dashed lines indicated columns or rows that have been struck off after their demand or supply gets exhausted.

Table 3.6.3.1

	Demand centers							
Supply centers	***D1***	***D2***	***D3***	***D4***	***Total Supply of each center***	***r1***	***r2***	***r3***
S1	3 (1000)	2 (4000)	7	6	***5000 / (1000) / (0) (ii)***	1	3	
S2	7 (2500)	5	2 (2000)	3 (1500)	***6000 / (0) (iv)***	1	1	1
S3	2 (2500)	5	4	5	***2500 / (0) (iii)***	2	2	2
Total Demand of each center	***6000 / (5000) / (2500) / (0)***	***4000 / (0) (i)***	***2000 / (0)***	***1500 / (0)***				

8

c1	1	3	2	2	
c2	1		2	2	
c3	5		2	2	

Total transportation cost = 3*1000 + 2*4000 + 7*2500 + 2*2000 + 3*1500 + 2*2500

= 3000 + 8000 + 17500 + 4000 + 4500 + 5000

= Rs.42,000.

Step 3: *Test of optimality:*

Optimality test is applied to find whether the initial basic feasible solution (*ibfs*) found by applying any of three methods discussed above is the best solution or can cost be reduced further. In *ibfs* transportation cost was found after allocating number of units to be transported to various demand centers to different cost cells. A new solution can be found by allocating certain units to different cost cell and then evaluating whether new cost is less than *ibfs* cost. Now there are numerous unallocated cells which can be used for allocation in new solution. That is why method of finding solution is an iterative process which involves:

- identifying new incoming cost cell.
- allocating units to this new cell and adjusting allocations to other occupied cells.
- identifying the outgoing cell.

3.6.4 ***Modified Distribution or MODI method*** is used for identification of incoming cell. This method selects that cell as incoming which in final solution will reduce the cost to the maximum. Feasibility of solution should be checked by applying m+n-1 rule before finding new solution by using MODI method.

This rule says that number of allocated cells should be equal to m+n-1 where 'm' is number of rows and 'n' is number of columns.

Number of allocated cells = 6

Number of rows 'm' = 3

Number of columns 'n' = 4

m+n-1 = 3+4-1 = 6

As condition is fulfilled so we can apply MODI method as discussed below.

Illustration of MODI method is applied on example shown in Table 6.6.1 and *ibfs* being found by using least cost method as shown in table 6.6.1.1 which has reproduced below. The following table only shows allocated cost cells.

Table 3.6.1.1					
	Demand centers				
Supply centers	***D1***	***D2***	***D3***	***D4***	***Total Supply of each center***

S1	3 (1000)	2 (4000)	7	6	***5000***
S2	7 (2500)	5	2 (2000)	3 (1500)	***6000***
S3	2 (2500)	5	4	5	***2500***
Total Demand of each center	***6000***	***4000***	***2000***	***1500***	

(i) Define each row of cost matrix as u_i and each column as v_j. In this example there are three rows so rows are defined as u_1, u_2 and u_3. Four columns are defined as v_1, v_2, v_3 and v_4.

(ii) Mark the cost of each occupied cell as sum of two intersecting defined variables as shown below:

For S1---D1: $u_1 + v_1 = 3$

S1---D2: $u_1 + v_2 = 2$

S2---D1: $u_2 + v_1 = 7$

S2---D3: $u_2 + v_3 = 2$

S2---D4: $u_2 + v_4 = 3$

S3---D1: $u_3 + v_1 = 2$

(iii) As total number of variables is seven and we have only six equations so put any one of the variables as zero. In most cases u_1 is assigned zero. After putting u_1 as zero calculate values of other variables which are:

$u_1 = 0$, $u_2 = 4$, $u_3 = -1$, $v_1 = 3$, $v_2 = 2$, $v_3 = -2$ and $v_4 = -1$.

(iv) Find out net evaluation cost of unoccupied cells by subtracting actual cost with sum of estimated costs represented by u and v variables. These has been calculated as below and shown in table 6.6.4.1 :

For unoccupied cell a_{13}: net evaluation cost = 7 – (0-2) = 9

a_{14}: net evaluation cost = 6 – (0-1) =7

a_{22}: net evaluation cost = 5 – (4+2) = -1

a_{32}: net evaluation cost = 5 – (-1+2) = 4

a_{33}: net evaluation cost = 4 – (-1-2) = 7

a_{34}: net evaluation cost = 5- (-1-1) = 7

Table 3.6.4.1					
	Demand centers				
	$v_1 = 3$	***$v_2 = 2$***	***$v_3 = -2$***	***$v_4 = -1$***	
$u_1 = 0$	3 (1000)	2 (4000)	7 ***[9]***	6 ***[7]***	***5000***

u_2 = 4	7 (2500)	5 ***[-1]***	2 (2000)	3 (1500)	***6000***
u_3 = -1	2 (2500)	5 ***[4]***	4 ***[7]***	5 ***[7]***	***2500***
	6000	***4000***	***2000***	***1500***	

Net evaluation of all unoccupied cells except of a_{22} comes out to be positive which indicates that introduction of any of this cost cell in new solution would increase the cost. As our objective is to reduce the cell with maximum negative quantum will be introduced into new solution. In table 6.6.4.1 only cell a_{22} is negative with net evaluation cost of -1. This implies that if this cell is introduced in new solution then cost will reduce by Rs.1 per unit. Thus, MODI method has identified incoming cell of new solution.

Now next step is allocate maximum number of units to this incoming cell. ***Stepping stone method*** is used to identify number of units to be allocated to cell a_{22}. This will also help to deduce the adjustments made to allocations of other occupied cells. For instance, if one unit is allocated to cell a_{22} then total units allocated to demand center D2 would be 4001 but is actual demand is 4000 units. So, one unit has to be deducted from other occupied cell. Similarly, assignment of one unit to a_{22} would increase total supply from 6000 to 60001 units. To maintain supply to 6000 units only we have to deduct one unit from other occupied cells in row S2.

To fulfill above mentioned objectives Stepping stone method involves following steps:

(i) From selected incoming cell either move horizontally or vertically to the nearest occupied cell. Purpose is to make a closed path in such a manner that at every corner there is an occupied cell starting from new incoming cell. The path should not have any diagonal movements.

(ii) The starting point i.e. incoming cell is marked as (+) and then next corner as (-) and this is repeated alternatively. (+) sign indicates that allocation will increase and (-) sign indicates that allocation will decrease by decided amount. The closed path is shown in table 6.6.4.2

Table 6.6.4.2

	Demand centers				
	D1	***D2***	***D3***	***D4***	
S1	3 (1000) ***(+)***	2 (4000) ***(-)***	7	6	***5000***
S2	7 (2500) ***(-)***	5 ***(+)***	2 (2000)	3 (1500)	***6000***

S3	2 (2500)	5	4	5	***2500***
	6000	***4000***	***2000***	***1500***	

(iii) To determine the maximum units allocated to the incoming cell identify *minus sign cell with smallest amount allocated.* In the designated closed path cell a12 and a21 have minus sign with 4000 and 2500 units as allocations respectively. As 2500 units is the smallest amount of units with minus sign so this amount would be added to the (+) sign cell and be subtracted from (-) sign cell. This would lead to new solution with allocations as shown in table 6.6.4.3.

Table 6.6.4.3					
	Demand centers				
	D1	***D2***	***D3***	***D4***	
S1	3 (2500)	2 (1500)	7	6	***5000***
S2	7	5 (2500)	2 (2000)	3 (1500)	***6000***
S3	2 (2500)	5	4	5	***2500***
	6000	***4000***	***2000***	***1500***	

Step 4: Lastly, to find whether this solution is best or can it be iterated further to get another optimal solution optimality of new solution by applying u-v method is done again.

Doing for table 3.6.4.2

For S1---D1: $u_1 + v_1 = 3$

S1---D2: $u_1 + v_2 = 2$

S2---D2: $u_2 + v_1 = 5$

S2---D3: $u_2 + v_3 = 2$

S2---D4: $u_2 + v_4 = 3$

S3---D1: $u_3 + v_1 = 2$

Putting $u_1 = 0$ we get

$u_1 = 0$, $u_2 = 3$, $u_3 = -1$, $v_1 = 3$, $v_2 = 2$, $v_3 = -1$ and $v_4 = 0$

Net evaluation cost for unoccupied cells would be:

For unoccupied cell a_{13}: net evaluation cost = 8

a_{14}: net evaluation cost = 6

a_{21}: net evaluation cost = 5

a_{32}: net evaluation cost = 4

a_{33}: net evaluation cost = 7

a_{34}: net evaluation cost = 6

As net valuation cost of all unoccupied cells is positive so cost cannot be reduced further. Thus, total transportation cost calculated from table 6.6.4.3 would be:

Total transportation cost = 3*3500 + 2*1500 + 5*2500 + 2*2000 + 3*1500 + 2*2500

= 10500 + 3000 + 12500 + 4000 + 4500 + 5000

= Rs.39,500.

So, it can be seen that cost has reduced from Rs.42,000 to Rs.39,500.

Exercise 2

1. A transportation problem is considered to be feasible if:
   (a) total demand is equal to total supply
   (b) number of allocated cells is equal to m+n-1
   (c) both
2. Modified Distribution (MODI) method is used to find:
   (a) incoming cost cell for new solution
   (b) adjustments in allocations of other occupied cells
   (c) both
3. Steeping stone method is used to find:
   (a) allocations for new incoming cell
   (b) outgoing cell
   (c) both

3.7 <u>DEGENERATE TRANSPORTATION MODEL</u>

Degeneracy occurs in a transportation model if number of allocations after finding initial basic feasible solution is not equal to m+n-1 where 'm' is number of rows and 'n' is number of columns. Resolution of degeneracy is illustrated in transportation model shown in table 6.7.1.

Table 6.7.1

	Demand centers			
Supply centers	**D1**	**D2**	**D3**	***Total Supply of each center***
S1	3	6	7	60
S2	8	5	7	30

S3	4	9	11	30
Total demand of each center	35	55	30	

Step 1: Check whether Total demand = total supply

35 + 55 + 30 = 60 + 30 + 30

120 = 120

Step 2: By using least cost method an initial basic feasible solution has been found as shown in table 6.7.1.1

Table 6.7.1.1

	Demand centers			
Supply centers	**D1**	**D2**	**D3**	***Total Supply of each center***
S1	3 (35)	6 (25)	7	60
S2	8	5 (30)	7	30
S3	4	9	11 (30)	30
Total demand of each center	35	55	30	

Total transportation cost = 3*35 + 6*25 + 5*30 + 11*30

= Rs.735

Step 3: *Optimality test:*

Number of allocations = 4

m + n – 1 = 3 + 3 – 1 = 5

As number of allocations is not equal to m+n-1 so solution is degenerate.

Consequences of degeneracy:

By applying u-v method:

$u_1 + v_1 = 3$

$u_1 + v_2 = 6$

$u_2 + v_2 = 5$

$u_3 + v_3 = 11$

By putting u_1 as zero, we got $v_1 = 3$, $v_2 = 6$ and $u_2 = -1$ but cannot solve for u_3 and v_3. In such cases MODI method to identify new incoming cell cannot be applied.

Resolution of degeneracy:

To find values of remaining variable assign minimum cost of zero to an unoccupied cell which would help to find values of remaining variables (in this case u_3 and v_3). This unoccupied cell is termed as "artificially occupied cell". In our example by treating cell a_{23} as artificially occupied cell help us to find values of u_3 and v_3.

So, another equation $u_2 + v_3 = 0$ would be formulated and then u_3 and v_3 can be calculated.

Thus, values of u_3 and v_3 would be 3 and 8 respectively as shown in table 6.7.2

Table 6.7.2				
	Demand centers			
	$v_1 = 3$	***$v_2 = 6$***	***$v_3 = 8$***	***Total Supply of each center***
$u_1 = 0$	3 (35)	6 (25)	7	60
$u_2 = -1$	8	5 (30)	7 (0)	30
$u_3 = 3$	4	9	11 (30)	30
Total demand of each center	35	55	30	

Net evaluation cost of unoccupied cell

For unoccupied cell a_{13}: net evaluation cost = -1

a_{21}: net evaluation cost = 6

a_{31}: net evaluation cost = -2

a_{32}: net evaluation cost = 0

So cell a_{31} is identified as incoming cell.

Closed loop:

Starting from incoming cell and moving vertically and horizontally a closed loop is made as shown in table 6.7.3. Also (+) and (-) signs are allocated alternatively starting from new incoming cell.

Table 6.7.3				
	Demand centers			
	$v_1 = 3$	***$v_2 = 6$***	***$v_3 = 8$***	***Total Supply of each center***
$u_1 = 0$	3 (35) ***(-)***	6 (25) ***(+)***	7	60
$u_2 = -1$	8	5 (30) ***(-)***	7 (0) ***(+)***	30
$u_3 = 3$	4 ***(+)***	9	11 (30) ***(-)***	30
Total demand of each center	35	55	30	

Looking at (-) sign cells minimum value is 30. Subtracting 30 from (-) sign cells and adding to (+) sign cells new allocations can be found. Thus, MODI method has helped to make adjustments to other occupied cells. However in this case 30 is subtracted from two cells namely a_{22} and a_{33}. One of these two cells would be treated as outgoing cell and other would be treated as artificially occupied cell with zero allocation. If both cells are treated as outgoing cells then the solution would become degenerate. And MODI method cannot be applied for next iteration. Thus, in case of tie of outgoing cell any one cell can be chosen as outgoing cell arbitrarily. In our example we have chosen cell a_{33} as outgoing cell and cell a_{22} as artificially occupied cell. New allocations are shown in table 6.7.4

Table 6.7.4				
	Demand centers			
	D1	**D2**	**D3**	***Total Supply of each center***
S1	3 (5)	6 (55)	7	***60***
S2	8	5 (0)	7 (30)	***30***
S3	4 (30)	9	11	***30***
Total demand of each center	***35***	***55***	***30***	

Total transportation cost = 3*5 + 5*55 + 5*0 + 7*30 + 4*30

= 15 + 275 + 0 + 210 + 120

= Rs.620

Step 4: *Iteration to new optimal solution:*

Number of allocations as shown in table 6.7.4 was equal to m+n-1. So MODI method can be applied to find a new optimal solution.

Application of u-v method and calculation of net evaluation costs is shown in table 6.7.5

Table 6.7.5				
	Demand centers			
	$v_1 = 3$	$v_2 = 6$	$v_3 = 8$	***Total Supply of each center***
$u_1 = 0$	3 (5)	6 (55)	7 ***[-1]***	60
$u_2 = -1$	8 ***[6]***	5 (0)	7 (30)	30
$u_3 = 1$	4 (30)	9 ***[2]***	11 ***[2]***	30
Total demand of each center	35	55	30	

So cell a_{13} becomes the new incoming cell.

Closed loop:

Starting from incoming cell and moving vertically and horizontally a closed loop is made as shown in table 6.7.6. Also (+) and (-) signs are allocated alternatively starting from new incoming cell.

Table 6.7.6				
	Demand centers			
	$v_1 = 3$	***$v_2 = 6$***	***$v_3 = 8$***	***Total Supply of each center***
$u_1 = 0$	3 (5)	6 ***(-)*** (55)	7 ***(+)***	60
$u_2 = -1$	8	5 (+) ***(0)***	7 (-) ***(30)***	30
$u_3 = 1$	4	9	11	30
Total demand of each center	35	55	30	

Subtracting 30 from (-) sign cells and adding 30 to (+) sign cells would give new allocations as shown in table 6.7.7.

Table 6.7.7				
	Demand centers			
	D1	**D2**	**D3**	***Total Supply of each center***
S1	3 (5)	6 (25)	7 (30)	***60***
S2	8	5 (30)	7	***30***
S3	4 (30)	9	11	***30***
Total demand of each center	***35***	***55***	***30***	

Total Transportation cost = 3*5 + 6*25 + 7*30 + 5*30 +4*30

= Rs.645

Optimality of this solution is again checked:

Number of allocations as shown in table 6.7.7 was equal to m+n-1. So MODI method can be applied to find a new optimal solution.

Application of u-v method and calculation of net evaluation costs shown in table 6.7.8. Net evaluation cost of all unoccupied cells comes out to be positive indicating cost cannot be reduced further and this is the most optimal solution.

Table 6.7.8				
	Demand centers			
	$v_1 = 3$	***$v_2 = 6$***	***$v_3 = 7$***	***Total Supply of each center***
$u_1 = 0$	3 (5)	6 (25)	7 (30)	60
$u_2 = -1$	8	5	7	30

	[6]	(30)	***[1]***	
u_3 = 1	4 (30)	9 ***[2]***	11 ***[3]***	30
Total demand of each center	35	55	30	

3.8 UNBALANCED TRANSPORTATION PROBLEM

The first step in finding initial basic feasible solution of a transportation problem is to check whether supply from various centers equals the demand of multiple centers. If available supply and demand requirements are not equal then problem becomes an unbalanced one. This can be resolved by adding a dummy supply center if total supply is less than total demand or add a dummy demand center if total supply is more than total demand. As such a center is a dummy one so cost of transporting each unit would be least and assumed to be zero. After matching the supply and demand requirements the problem can be solved as basic transportation problem which have been discussed in detail in previous sections.

An illustration of unbalanced transportation problem is shown in table 6.8.1

Table 6.8.1				
	Demand centers			
Supply centers	**D1**	**D2**	**D3**	***Total Supply of each center***
S1	5	1	7	***10***
S2	6	4	6	***80***
S3	3	2	5	***15***
Total demand of each center	***75***	***20***	***50***	

Total Supply = 10 + 80 + 15 = 105 units

Total Demand = 75 + 20 + 50 = 145units

As total demand is more than total supply so a dummy supply center is added to fulfill additional demand shown in table 6.8.2.

Table 6.8.2				
	Demand centers			
Supply centers	**D1**	**D2**	**D3**	***Total Supply of each center***
S1	5	1	7	***10***
S2	6	4	6	***80***
S3	3	2	5	***15***
S4	0	0	0	***40***
Total demand of each center	***75***	***20***	***50***	

An initial basic feasible solution is calculated by using VAM. The allocations are shown in table 6.8.3

Table 6.8.3

	Demand centers			
Supply centers	**D1**	**D2**	**D3**	***Total Supply of each center***
S1	5	1 (10)	7	***10***
S2	6 (60)	4 (10)	6 (10)	***80***
S3	3 (15)	2	5	***15***
S4	0	0	0 (40)	***40***
Total demand of each center	***75***	***20***	***50***	

Total transportation cost = Rs.515

Optimality test:

Number of allocations = 6 = m+n-1

Application of u-v method and calculation of net evaluation cost of unoccupied cells is shown in table 6.8.4

Table 6.8.4

	Demand centers			
	v_1	v_2	v_3	***Total Supply of each center***
u_1	5 ***[2]***	1 (10)	7 ***[4]***	***10***
u_2	6 (60)	4 (10)	6 (10)	***80***
u_3	3 (15)	2 ***[1]***	5 ***[2]***	***15***
u_4	0 ***[0]***	0 ***[2]***	0 (40)	***40***
Total demand of each center	***75***	***20***	***50***	

As net evaluation cost of all unoccupied cells was found to be either positive or zero so cost cannot be reduced further and solution found is most optimal.

3.9 MAXIMIZATION TRANSPORTATION PROBLEM

Transportation problems can also be used to maximize the objective of increasing profits or revenues. In such cases problem involves value of profit or revenues in cells obtained from transporting one unit from one supply center to a particular demand center. First step in solving such problems is to convert profit matrix into cost matrix by subtracting the highest profit value in the entire matrix from all other cells. The resultant matrix would be a cost matrix which can be solved by using basic transportation problem methodology. It is important to remember that after finding optimal solution one should convert that optimal cost into maximum profit by multiplying profit cell values with the found allocated cells.

An illustration has been provided below in table 6.9.1.

Table 6.9.1					
	Demand centers				
Supply centers	*D1*	*D2*	*D3*	*D4*	***Total Supply of each center***
S1	6	6	6	4	***1000***
S2	4	2	4	5	***700***
S3	5	6	7	8	***900***
Total Demand of each center	***900***	***800***	***500***	***400***	

Step 1: Demand = Supply = 2600 units, so problem is a balanced one.

Step 2: Convert profit matrix into cost matrix by subtracting largest value i.e. 8 from all other values. The cost matrix is shown in Table 6.9.2

Table 6.9.2					
	Demand centers				
Supply centers	*D1*	*D2*	*D3*	*D4*	***Total Supply of each center***
S1	2	2	2	4	***1000***
S2	4	6	4	3	***700***
S3	3	2	1	0	***900***
Total Demand of each center	***900***	***800***	***500***	***400***	

By applying VAM following allocation table was obtained (Table 6.9.3)

Table 6.9.3					
	Demand centers				
Supply centers	*D1*	*D2*	*D3*	*D4*	***Total Supply of each center***
S1	2 (200)	2 (800)	2	4	***1000***
S2	4 (700)	6	4	3	***700***
S3	3	2	1 (500)	0 (400)	***900***
Total Demand of each center	***900***	***800***	***500***	***400***	

Step 3: *Optimality test*

Number of allocations = 5 is not equal to m+n-1 = 6. Thus problem is of degeneracy. As discussed in section 6.7 degeneracy can be resolved by creating an artificial occupied cell which would help in applying MODI method. In this case cell a_{32} is created as an artificial occupied cell and zero units are allocated to it.

So following equations would be formulated:

$u_1 + v_1 = 2$; $u_1 + v_2 = 2$; $u_2 + v_1 = 4$; $u_3 + v_2 = 2$; $u_3 + v_3 = 1$; $u_3 + v_4 = 1$

By putting v1 = 0 we found values of other variables as shown in table 6.9.4. This table also shows net evaluation cost of unoccupied cells.

Table 6.9.4					
	Demand centers				
	$v_1 = 0$	***$v_2 = 0$***	***$v_3 = -1$***	***$v_4 = -1$***	***Total Supply of each center***
$u_1 = 2$	2 (200)	2 (800)	2 ***[1]***	4 ***[3]***	***1000***
$u_2 = 4$	4 (700)	6 ***[2]***	4 ***[1]***	3 ***[0]***	***700***
$u_3 = 2$	3 ***[1]***	2 ***[0]***	1 (500)	0 (400)	***900***
Total Demand of each center	***900***	***800***	***500***	***400***	

As net evaluation costs of all unoccupied cells was found to be either positive or zero so it can be concluded that obtained solution is most optimal.

Now it is to remember that we have converted profit matrix into cost matrix and objective was to find maximum profit. So, multiple profit values of allocated cells with number of units in those cells. Thus,

Total Profit = 6*200 + 6*800 + 4*700 +7*500 + 8*400

= Rs. 15,500.

Exercise 3

1. Degeneracy occurs when all destinations cannot be allocated number of units required from various sources.

   (a) True (b) False

2. Degeneracy can be resolved by creating an artificially occupied cell which is allocated zero units.

   (a) True (b) False

3. Dummy resources are used to remove mismatch between number of units available and required.

   (a) True (b) False

4. Significant transportation cost is involved in transporting units from dummy source to a particular destination.

   (a) True (b) False

3.10 SUMMARY

Transportation problem is a special type of linear programming problem which is applied to fulfill the objective of finding most optimal minimum cost of transporting specified number of units of a particular product available from multiple known sources and to be distributed to multiple known destinations. The main purpose of transportation problem is minimization of transportation cost. But a version of model can also be used for finding the maximum objective a firm can achieve in transporting fixed number of units. The chapter discusses various assumptions required to implement transportation model. In case of violation of certain assumptions such as equality of demand and supply and degeneracy special models have also been discussed in detail. All these models require three elements namely: cost per unit of transportation from a particular supply center to a particular demand center, capacity of each supplier and demand of each destination to complete the structure. Various methods such as least cost, north-west corner and Vogel's approximation have been discussed to find initial basic feasible solution. The illustrations indicate least cost and Vogel's method to be better method of finding initial solution. Optimality of solution or to test whether a better solution can be found was found by using Modified Distribution method. Lastly, each solution goes through repeated process of iteration to conclude a final and most optimal solution.

3.11 GLOSSARY

- ***Sources:*** are limited and known numbers of supply centers which supply a limited amount of product.
- ***Destinations:*** are limited and known numbers of demand centers which source from various supply centers according to their requirement.
- ***Feasibility:*** A transportation model is considered to be feasible if total supply of all sources is equal to total demand of all destinations.
- ***Initial basic feasible solution:*** is the first feasible solution found by applying least cost, north-west or Vogels' approximation method.
- ***MODI method:*** is also known as u-v method. It is used to check the optimality of initial solution by identifying new incoming cell and adjusting the allocations of other occupied cells.
- ***Net evaluation cost:*** is the difference between estimated and original cost of each unoccupied cell. A negative net evaluation cost cell would indicate the cell to be an incoming cell having potential to reduce cost further.
- ***Dummy:*** When there is a mismatch between available and required number of units then to make model a feasible one dummy resources are added.
- ***Degeneracy:*** When number of allocated cells is not equal to m+n-1 condition then MODI method cannot be applied leading to degeneracy.

3.12 ANSWERS TO CHECK YOUR PROGRESS/ SELF ASSESSMENT EXERCISE

Exercise 1:

1. A
2. C
3. C

Exercise 2

1. C
2. C
3. C

Exercise 3

1. A
2. A
3. A
4. B

3.13 REFERENCES/ SUGGESTED READINGS

- Hillier, F.S. and Lieberman, J. G., *Operations Research,* Tata McGraw Hill, 2009, 10th Reprint.
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- Natarajan, A.M., Balasubramani, P. and Tamilarasi, A., *Operations Research,* Pearson, 2012, 9th Edition.

3.14 TERMINAL AND MODEL QUESTIONS

1. What are the assumptions of a transportation problem? Explain the ways of resolving if assumptions are violated.
2. What is degeneracy? How does the problem of degeneracy arise in a transportation problem?
3. Consider the transportation problem having the following parameter table:

Source	**Destination**				
	1	**2**	**3**	**4**	**Supply**
1	3	7	6	4	***5***
2	2	4	3	2	***2***
3	4	3	8	5	***3***
Demand	***3***	***3***	***2***	***2***	

Use both Northwest corner rule and Vogel's approximation method to obtain an initial basic feasible solution.

4. Consider the transportation problem having the following parameter table:

Source	**Destination**				
	1	**2**	**3**	**4**	**Supply**
1	500	600	400	200	***10***
2	200	900	100	300	***20***
3	300	400	200	100	***20***
4	200	100	300	200	***10***
Demand	***20***	***10***	***10***	***20***	

Obtain an optimal solution by using least cost and north-west corner method to find initial solution.

Chapter 4: ASSIGNMENT MODEL

4.0 OBJECTIVES

The students should be able to understand:

- Concept of assignment problem
- Assumptions of assignment problem
- Hungarian method of solving assignment problem
- Variants of assignment problem namely unbalanced and maximization

4.1 INTRODUCTION

Assignment model is a special form of transportation problem. The objective of assignment model is to allocate or assign jobs to machines or jobs to people. Most common application of assignment model is assigning various jobs to various limited employees. This objective can be fulfilled to achieve based on certain criteria. Various workers can be assigned to various jobs based on the criteria of least time taken to perform the job or least cost incurred to the company. Assignment model has a special application in location decision making. Which department should be assigned to which building in a University? Grocery store should be located where within a supermarket? In a mall an apparel store should locate itself at which part of the mall? Assignment model can be helpful in solving such location problems.

Assignment problem is similar to transportation problem in two aspects: firstly, both models have two players a limited and known number of suppliers or employees and limited and know number of demand centers or jobs. Secondly, in case of transportation allocation of certain number of units to a particular demand center was primarily based on fundamental principle of minimum cost required to transport one unit. Similarly, in case of assignment problem assigning of one job to a particular employee is based on certain criteria of either minimizing cost or minimizing time to do specific job. But both models have one major difference. In case of transportation problem different supply and demand centers can have different capacity of output and requirements respectively. Whereas in case of assignment problem all supplies and demands equal to one. This is because of fundamental of assignment problem i.e. one worker can be assigned to do only job at a time, one geographic area can be assigned only one store or department and so on.

4.2 <u>PARAMETERS AND STRUCTURE OF TRANSPORTATION MODEL</u>

An assignment model specifically includes following parameters:

- Limited number of jobs.
- Limited number of employees.
- Known cost of performing a particular job by a particular employee.

These three elements are shown in the Table 4.2.1 indicating generalized structure of a transportation model.

Table 4.2.1

	Jobs		
Employees	***J1***	***J2***	***J3***
E1	a_{11}	a_{12}	a_{13}
E2	a_{21}	a_{22}	a_{23}
E3	a_{31}	a_{32}	a_{33}

Table 4.2.1 illustrates:

- Three employees: E1, E2, and E3.
- Three jobs: J1, J2 and J3.
- Cost of performing a job by a specific employee. For instance, a_{11} is the cost of performing job J1 by employee E1.

One important aspect that needs to be noted in the given table is that each job can be performed by all the given employees. This is most likely to happen in case of service organizations such as banks, post offices, retail stores etc. in a bank an employee can perform various jobs so a manager has to face with a situation where he/she has to assign a particular job to an employee. Thus, assignment problem of assigning employee to jobs finds its application mostly in case of service organizations which follow batch production system

Another important aspect that needs to be taken into consideration is the criteria on the basis of which assignments have to be made. In above illustration values in the cells represented cost of performing a particular job by an employee. So, if objective is to minimize cost manager would assign a job to an employee who performs it at the least cost. Similarly if the problem is allocation of departments to various buildings then criteria could be minimum time required to access similar departments. For sake of another illustration various machines can be located at different locations on a shop floor. In such cases criteria can be material handling costs for these machine or work flow between them. The objective would be to minimize material handling costs or minimize time between machines which have highest work flow between them.

4.3 ASSUMPTIONS OF ASSIGNMENT PROBLEM

The application of assignment problem can be formulated if it satisfies following assumptions:

- Number of assignees is equal to number of tasks. If there are three workers then there should be three jobs to which these workers can be assigned.
- Each assignee or employee can be assigned to only one job.
- Each task is to be performed by only one assignee or employee.
- There is a cost associated with a particular worker accomplishing a particular job.
- The objective is to determine a way in which employees can be assigned jobs so that cost is minimum.

Exercise 1

1. Assignment problem is predominantly applied in making____________decisions.
2. Which of the following is not an assumption of assignment problem?
   (a) Each assignee or employee can be assigned to only one job.
   (b) Each task is to be performed by only one assignee or employee.
   (c) All the employees can perform all the jobs
   (d) Criteria for assignment is proportional to number of employees
3. Similarity between assignment and transportation problem is:
   (a) Both deal with cost matrix
   (b) Objective is to minimize cost or maximize profit
   (c) Source and destination are known and limited.
   (d) All of the above
4. Assignment problems majorly find their application as a quantitative tool for location decision of machines in case of which of the following production system
   (a) Batch production systems
   (b) Mass production systems

4.4 ALGORITHM FOR ASSIGNMENT PROBLEM

It has been discussed that assignment problem is a special case of transportation problem. Because of its difference of number of jobs and employees being equal to one a special type of algorithm known as ***Hungarian***

algorithm is used for solving assignment problems. The steps involved are discussed by taking following illustration:

Example 4.4.1: Table 4.4.1 shows estimated job completion time in days of three jobs by three employees. The objective is to assign employees to given jobs in such a way that total job completion time is minimized.

Table 4.4.1			
	Jobs		
Employees	***J1***	***J2***	***J3***
E1	10	15	9
E2	9	18	5
E3	6	14	3

Step 1: Check if number of rows is equal to number of columns to find whether problem has a balanced matrix.

In this case number of rows = number of columns = 3

Step 2: From each row subtract the minimum cell value (in this case time) from all other values of that row. In row 1 represented by E1 minimum value is 9 which is subtracted from 10 and 15.

This would give values as shown in table 4.4.2

Table 4.4.2			
	Jobs		
Employees	***J1***	***J2***	***J3***
E1	1	6	0
E2	4	13	0
E3	3	11	0

Step 3: From each column subtract the minimum cell value (in this case time) from all other values of that column. In column 1 represented by J1 minimum value is 1 which is subtracted from 4 and 3.

This would give values as shown in table 4.4.3

Table 4.4.3			
	Jobs		
Employees	***J1***	***J2***	***J3***
E1	0	0	0
E2	3	7	0
E3	2	5	0

Step 4: *Make assignments:*

(i) Starting from first row make assignment or allocations in the row which has only one zero. This would imply assigning an employee to a particular job. First row has three zeros so assignment cannot be done. Moving to second row it has only one zero so assign J3 to E2 *(indicated by zeroes in brackets)*. As J3 has been assigned so it cannot be further assigned to any other employee so any other zeroes in J3 would be struck off *(indicated by (×))* as shown in table 4.4.4. Row 3 has no zeros left as zero in J3 has been struck off.

Table 4.4.4			
	Jobs		
Employees	***J1***	***J2***	***J3***
E1	0	0	0 *(×)*
E2	3	7	***(0)***
E3	2	5	0 *(×)*

(ii) Same procedure is adopted by column wise. In first column there is only one zero so assignment can be made. Thus, J1 is assigned to E1. As E1 cannot be assigned any other job so any more zeros (if any) in that row would be struck off as shown in table 4.4.5. Moving to column 2 represented by J2 no zeros are left so no assignment can be made. Column 3 does not have any free zeros except assigned and crossed zeros.

Table 4.4.5			
	Jobs		
Employees	***J1***	***J2***	***J3***
E1	***(0)***	0 *(×)*	0 *(×)*
E2	3	7	***(0)***
E3	2	5	0 *(×)*

Step 5: *Check for optimality:*

An assignment solution is said to be optimal if all jobs are assigned. In this case only two jobs i.e. J1 and J3 have been assigned to E1 and E2 respectively. Also steps involved in checking optimality of solution have been discussed below:

(i) Mark the row (√) in which assignment has not been made. In this case it would be row 3 represented by E3. It would be called as unassigned row.

(ii) Look for any crossed zero in that row. Row 3 has a crossed zero in cell a_{33}. Most likely an unassigned row would have a crossed zero because of row and column reduction procedure.

(iii) Mark such a column (√). In this case that column would be J3.

(iv) In that column find the cell where assignment has been made. In this it would be cell a_{23}. Mark that row (√). It would be row 2 represented by E2.

(v) Repeat the above steps.

(vi) Draw lines through unmarked rows and marked columns. If number of marked lines is equal to number of assignments then the solution has reached an optimal point.

Table 4.4.6				
	Jobs			
Employees	***J1***	***J2***	***J3***	
E1	***(0)***	0 *(×)*	0 *(×)*	
E2	3	7	***(0)***	(√) ***(iv)***
E3	2	5	0 *(×)*	(√) ***(i)***
			(√) ***(iii)***	

In this case there was only one unassigned row so in the end there would be only two marked lines as shown in Table 4.4.6. This implies solution is not optimal.

Step 6: *Iterate for optimal solution:*

Following steps needs to be followed to iterate towards an optimal solution.

(i) From the solution shown in table 4.4.6 examine the cells that do not lie on the marked lines. These are termed as unoccupied cells

(ii) Select the minimum value from these cells. In this case it would be 2

(iii) Subtract this minimum value from all of the unoccupied cells.

(iv) Add this smallest value to the value at intersection cell. In this case intersection cell has 0.

These steps are shown in table 4.4.7

Table 4.4.7			
	Jobs		
Employees	***J1***	***J2***	***J3***
E1	***(0)***	0	2
E2	1	5	***(0)***
E3	0	3	0

This gives next feasible solution. Assignments and optimality test would again be applied on this solution. The application of assignments is shown in table 4.4.8

Table 4.4.8			
	Jobs		
Employees	***J1***	***J2***	***J3***
E1	0 *(×)*	***(0)***	2
E2	1	5	***(0)***
E3	***(0)***	3	0 *(×)*

As all the jobs are assigned and there is no unassigned row or column so obtained solution is optimal.

Assignments have done as:

Employee	**Job**	**Time taken to complete the job**
E1	J2	15
E2	J3	5
E3	J1	6
Total time		**26 days**

Example 4.4.2: A firm needs to assign three project leaders to three projects. Based on different backgrounds and experiences of the leaders various leader-client assignments differ in terms of projected completion times. The possible assignments and estimated compeltio0n times in days are given in table 4.4.9. Estimate minimum total completion times by assigning projects to respective clients.

Table 4.4.9			
	Client		
Project leader	***C1***	***C2***	***C3***
A	10	16	32
B	14	22	40
C	22	24	34

Step 1: Check if number of rows is equal to number of columns to find whether problem has a balanced matrix.

In this case number of rows = number of columns = 3

Step 2: Row reduction and Column reduction will give results as shown in table 4.4.10

Table 4.4.10	
	Client

Project leader	***C1***	***C2***	***C3***
A	0	4	10
B	0	6	4
C	0	0	0

Step 3: Project leaders would be assigned clients by adopting the process as discussed. Proper assignments are shown in table 4.4.11

Table 4.4.11			
	Client		
Project leader	***C1***	***C2***	***C3***
A	***(0)***	4	10
B	*0 (×)*	6	4
C	*0 (×)*	***(0)***	*0 (×)*

Step 4: As there are only two assignments made instead of requirement of three so solution is not optimal. To find next optimal solution rows and columns were marked by adopting the discussed process. The results have been shown in table 7.4.12

Table 7.4.12				
	Client			
Project leader	***C1***	***C2***	***C3***	
A	***(0)***	4	10	√
B	*0 (×)*	6	4	√
C	*0 (×)*	***(0)***	*0 (×)*	
	√			

Table 4.4.13			
	Client		
Project leader	***C1***	***C2***	***C3***
A	0	0	6
B	0	2	0

C	4	0	0

Now as shown in table 4.4.13 number of zeros is more than number of occupied cells implying that multiple assignments can be made to a particular project leader. In this case there can be possible two solutions shown in table 4.4.14 (a) and 4.4.15 (b).

<table>
<tr><th colspan="4">Table 7.4.14 (a)</th><th rowspan="6"></th><th colspan="3">Table 4.4.15 (b)</th></tr>
<tr><th></th><th colspan="3">Client</th><th colspan="3">Client</th></tr>
<tr><th>Project leader</th><th>C1</th><th>C2</th><th>C3</th><th>C1</th><th>C2</th><th>C3</th></tr>
<tr><td>A</td><td>(0)</td><td>0 (×)</td><td>6</td><td>0 (×)</td><td>(0)</td><td>6</td></tr>
<tr><td>B</td><td>0 (×)</td><td>2</td><td>(0)</td><td>(0)</td><td>2</td><td>0 (×)</td></tr>
<tr><td>C</td><td>4</td><td>(0)</td><td>0 (×)</td><td>4</td><td>0 (×)</td><td>(0)</td></tr>
<tr><td colspan="8"></td></tr>
<tr><td colspan="4">Assignments are made as</td><td colspan="4">Assignments are made as</td></tr>
<tr><td>A--- C1</td><td colspan="3">10</td><td colspan="3">A--- C2</td><td>16</td></tr>
<tr><td>B--- C3</td><td colspan="3">40</td><td colspan="3">B--- C1</td><td>14</td></tr>
<tr><td>C--- C2</td><td colspan="3">24</td><td colspan="3">C--- C3</td><td>34</td></tr>
<tr><td>Total completion time</td><td colspan="3">74 days</td><td colspan="3">Total completion time</td><td>64 days</td></tr>
</table>

As total completion time of 64 days is less than 74 days so assignments as shown in table 7.4.13 (b) would be concluded.

4.5 <u>VARIATIONS OF ASSIGNMENT PROBLEM</u>

Unbalanced Assignment problem: occurs when number of employees is not equal to number of jobs. In case of mismatch between jobs and number of employees assignment problem is resolved by adding dummy resources. If number of employees is more than available jobs then extra employee would sit idle and add cost to the company. In this case a dummy column representing an imaginary job would be added to matrix to solve by Hungarian method. In the end extra employee would be assigned this non-existent job. So he/she has to wait for availability of job. If jobs are more than employees then a job remains unassigned and might delay the entire project. In this case a dummy employee would be added to the matrix. In the end this job would be assigned a non-existent employee and it would have to wait for availability of an employee before it gets implemented.

Maximization problem: Sometimes the problem is to assign jobs to employees in such a manner so as to maximize profits or revenues. For instance if a store manager has to decide location of new store in a mall where he/she has multiple location choices then mange would tend to select a location which provides maximum sales or revenues. In such cases criteria of assigning is maximization of a parameter like sales. Such assignment problems can be solved by converting maximization problem into minimization problem. This is done by subtracting all the values in the profit matrix from the largest value. This would give values represented as opportunity losses instead of profits. Now such a matrix can be solved by general Hungarian method.

Exercise 2

1. Unbalanced assignment problem can be solved by:
   (a) adding dummy row
   (b) adding dummy column
   (c) cannot be solved
   (d) both a and b
2. The cell values in dummy row or column represents:
   (a) zero assignment value
   (b) non-assignment of jobs to employees
   (c) both a and b

4.5.1 UNBALANCED ASSIGNMENT PROBLEM

Table 7.5.1.1 shows project completion times where four project leaders have to be assigned three available projects.

Table 4.5.1.1			
	Client		
Project leader	***C1***	***C2***	***C3***
A	10	15	9
B	9	18	5
C	6	14	3
D	8	16	6

As number of project leaders are more (i.e. 4) than available projects (i.e. 3) so a dummy project C4 has to be created to make solution feasible as shown in table 4.5.1.2

Table 4.5.1.2				
	Client			
Project leader	***C1***	***C2***	***C3***	***C4***

A	10	15	9	0
B	9	18	5	0
C	6	14	3	0
D	8	16	6	0

Row reduction and column reduction would give table 4.5.1.3

Table 4.5.1.3				
	Client			
Project leader	***C1***	***C2***	***C3***	***C4***
A	4	1	6	0
B	3	4	2	0
C	0	0	0	0
D	2	2	3	0

Make assignments: Table 4.5.1.4 shows assignments made

Table 4.5.1.4				
	Client			
Project leader	***C1***	***C2***	***C3***	***C4***
A	4	1	6	***(0)***
B	3	4	2	*0 (×)*
C	***(0)***	*0 (×)*	*0 (×)*	*0 (×)*
D	2	2	3	*0 (×)*

Optimal Solution:

Table 4.5.1.4					
	Client				
Project leader	***C1***	***C2***	***C3***	***C4***	
A	4	1	6	***(0)***	√
B	3	4	2	*0 (×)*	√
C	***(0)***	*0 (×)*	*0 (×)*	*0 (×)*	

D	2	2	3	*0 (×)*	√
				√	

From unmarked cells minimum value is 1 so subtract it from all unmarked cells and add to intersection value (table 4.5.1.5)

Table 4.5.1.5				
	Client			
Project leader	***C1***	***C2***	***C3***	***C4***
A	3	0	5	0
B	2	3	1	0
C	0	0	0	1
D	1	1	2	0

Make assignments:

Table 4.5.1.6				
	Client			
Project leader	***C1***	***C2***	***C3***	***C4***
A	3	***(0)***	5	*0 (×)*
B	2	3	1	***(0)***
C	***(0)***	*0 (×)*	*0 (×)*	1
D	1	1	2	*0 (×)*

Optimal Solution:

As number of assignments are still less (i.e. 3) than as required (i.e. 4) so next optimal solution would be as shown in table 4.5.1.7

Table 4.5.1.6					
	Client				
Project leader	***C1***	***C2***	***C3***	***C4***	
A	3	***(0)***	5	*0 (×)*	
B	2	3	1	***(0)***	√
C	***(0)***	*0 (×)*	*0 (×)*	1	

D	1	1	2	*0 (×)*	√
				√	

Minimum value from unoccupied cells is 1. Subtracting it from unoccupied cells and adding it to intersection values we get table 4.5.1.7

Table 4.5.1.7				
	Client			
Project leader	***C1***	***C2***	***C3***	***C4***
A	3	0	5	1
B	1	2	0	0
C	0	0	0	2
D	0	0	1	0

Making assignments: Now as shown in table 4.5.1.7 number of zeros is more than number of occupied cells implying that multiple assignments can be made to a particular project leader. In this case there can be possible two solutions shown in table 4.5.1.7 (a) and 4.5.1.7 (b).

Table 4.5.1.7 (a)						**Table 4.5.1.7 (b)**			
	Client					**Client**			
Project leader	***C1***	***C2***	***C3***	***C4***		***C1***	***C2***	***C3***	***C4***
A	3	***(0)***	5	1		3	***(0)***	5	1
B	1	2	***(0)***	*0 (×)*		1	2	*0 (×)*	***(0)***
C	***(0)***	*0 (×)*	*0 (×)*	2		*0 (×)*	*0 (×)*	***(0)***	2
D	*0 (×)*	*0 (×)*	1	***(0)***		***(0)***	*0 (×)*	1	0

Assignments are made as		***Assignments are made as***	
A--- C2	***15***	***A --- C2***	***15***
B--- C3	***5***	***B --- C4***	***0***
C--- C1	***6***	***C --- C3***	***3***
D--- C4	***0***	***D --- C1***	***8***
Total completion time	***26 days***	***Total completion time***	***26 days***

Thus, total project completion time comes out to be same for both variants of assignments i.e. 26 days. So, any of the assignment solution can be selected. The important aspect to be noted is that results according to table 4.5.1.7 (a) project leader D would sit idle whereas according to table 4.5.1.1 (b) project leader B would sit idle. Thus, it depend on management which solution to be adopted.

4.5.2 MAXIMIZATION ASSIGNMENT PROBLEM

A distributor distributes a product in four territories (P, Q, R, S) each of which has to be assigned a sales representative (A, B, C, D). The selected criteria for assignment were revenue generated by each sales representative in each territory. Estimated revenues from past experience in Rs.000s have been shown in table 4.5.2.1. The objective is to assign a territory to a particular sales representative so that maximum revenues can be generated.

Table 4.5.2.1

	Sales Territory			
Sales Representative	**P**	**Q**	**R**	**S**
A	44	80	52	60
B	60	56	40	72
C	36	60	48	48
D	52	76	36	40

Convert sales matrix of table 4.5.2.1 into cost matrix shown in table 4.5.2.2 by subtracting larges revenue from all other values.

Table 4.5.2.2

	Sales Territory			
Sales Representative	**P**	**Q**	**R**	**S**
A	36	0	28	20
B	20	24	40	8
C	44	20	32	32
D	28	4	44	40

<u>*Row reduction:*</u>

Table 4.5.2.3

	Sales Territory			
Sales Representative	**P**	**Q**	**R**	**S**

A	36	0	28	20
B	12	16	32	0
C	24	0	12	12
D	24	0	40	36

Column reduction:

Table 4.5.2.4				
	Sales Territory			
Sales Representative	**P**	**Q**	**R**	**S**
A	24	0	16	20
B	0	16	20	0
C	12	0	0	12
D	12	0	28	36

Make Assignments:

Table 4.5.2.5					
	Sales Territory				
Sales Representative	**P**	**Q**	**R**	**S**	
A	24	***(0)***	16	20	√
B	***(0)***	16	20	*0 (×)*	
C	12	*0 (×)*	***(0)***	12	
D	12	*0 (×)*	28	36	√
		√			

Next Optimal Solution: Subtract minimum unoccupied value i.e. 12 from unmarked cells and add it to the intersecting values.

Table 4.5.2.6				
	Sales Territory			
Sales Representative	**P**	**Q**	**R**	**S**
A	12	0	4	8
B	0	28	20	*0*
C	12	12	0	12
D	0	0	16	24

Make Assignments:

Table 4.5.2.7				
	Sales Territory			
Sales Representative	**P**	**Q**	**R**	**S**
A	12	***(0)***	4	8
B	*0 (×)*	28	20	***(0)***
C	12	12	***(0)***	12
D	***(0)***	*0 (×)*	16	24

So final assignments would be:

A ---Q = 80

B----S = 72

C--- R = 48

D ---P = 52

Total = 254

Thus, distributor can assign sales territory to respective sales representative in solved manner to maximize revenues.

4.6 SUMMARY

Assignment problem is a special form of linear programming problem which specifically deals with assigning jobs to employees, machines to workers, departments to locations etc. assignment problem can also be dealt in similar fashion as transportation problem. The only difference between transportation and assignment was that the demand and supply in case of assignment was equal to one whereas in transportation different sources and demand centers have different supply and requirements. Hungarian method has been discussed to solve assignment problem. Two variants namely unbalanced and maximization has also been discussed with illustrations.

4.7 GLOSSARY

- ***Assignment problem:*** deals with allocating or assigning jobs to employees.
- ***Unbalanced assignment problem:*** occurs when there is a mismatch between number of jobs and number of employees.
- ***Maximization assignment problem:*** occurs when criteria of assigning are sales, profits or revenues and objective is to maximize them.

4.8 ANSWERS TO CHECK YOUR PROGRESS/ SELF ASSESSMENT EXERCISE

Exercise 1

1. Answer: Location
2. D
3. D
4. A

Exercise 2

1. d
2. c

4.9 REFERENCES/ SUGGESTED READINGS

- Hillier, F.S. and Lieberman, J. G., *Operations Research,* Tata McGraw Hill, 2009, 10th Reprint.
- Anderson, D.R., Sweeney D. J. and Williams, T.A., *An Introduction to Management Science: Quantitative Approach to Decision Making,* South Western Cengage Learning, 11th Edition.
- Natarajan, A.M., Balasubramani, P. and Tamilarasi, A., *Operations Research,* Pearson, 2012, 9th Edition.

4.10 TERMINAL AND MODEL QUESTIONS

1. Show that the assignment problem is a special case of the transportation problem.
2. Discuss the similarity and difference in transportation and assignment model.
3. A company is faced with the problem of assigning 4 machines to 6 different jobs. The profits are estimated as follows:

	Machine			
Job	A	B	C	D
1	3	6	2	6
2	7	1	4	4
3	3	8	5	8
4	6	4	3	7
5	5	2	4	3
6	5	7	6	4

Solve the problem to maximize total profits

4. Find the proper assignment of the assignment problem whose cost matrix is given as under.

	1	***2***	***3***	***4***	***5***
A	10	6	4	8	3
B	2	11	7	7	6
C	5	10	11	4	8

D	6	5	3	2	5
E	11	7	10	11	7

5. Find the assignment of salesman to various districts which will yield maximum profit

	1	***2***	***3***	***4***
A	16	10	14	11
B	14	11	15	15
C	15	15	13	12
D	13	12	14	15

Chapter 5

Game Theory

Structure

5.1 Objectives

After reading the following sections the reader will be able:

- To understand the meaning of the game and need for game theory
- To know the relevance of the technique in real life
- To know the methods of solving Two person zero sum game

- To know the methods of solving 2*n and m*2 games
- To understand the limitations of Game theory

5.2 Introduction

Game Theory is applicable in the situations of decision making when two or more competitors are involved having conflicting interests. It is appropriate in a wide variety of contexts in which one must make a decision whose outcome will be determined by the actions of one or more individuals. If a firm and a union leader negotiate next year's wage policy, they are playing a game. A shopkeeper setting price of a product to be sold to customers is playing game with them.

5.3 Meaning

A *game* represents a competitive or conflicting situation between two or more *players*. A situation is termed as game when it has following characteristics:

- The number of players is finite.
- Each of the player has a finite number of possible courses of action(strategies)
- Players are having conflicting interests
- Rules of the game are specified and known to all players
- The outcome of the game is affected by the choices made by all the players

Thus in game theory, a player selects his moves without any knowledge of the moves chosen by the other players. The simultaneous choices of all players lead to the respective *payoffs* of the game. *Game theory* deals with the determination of the optimal strategies for each player. In game theory, all information concerning the game is known to both players; i.e., there is complete information. The players are said to be rational and intelligent. A rational person is one who acts in such a way as to maximize his or her expected payoff or utility. An intelligent person is one who can deduce what his or her opponent will do when acting rationally.

In sum, Game theory provides a systematic quantitative approach for analyzing competitive situations in which the competitors make use of logical processes and techniques in order to determine an optimal strategy for winning.

5.3.1 Assumptions

1. The players act rationally and intelligently.
2. Each player has relevant information.
3. Players take decisions independently.
4. The aim of every player is to optimize.

5. Each player can use the information in finite moves with finite choices for each move.
6. The pay-offs are fixed and known in advance.

5.3.2 Types of Games: There are various types of games as explained below:

- ***Two- person Game***: A situation, when number of players is two, though having many possible choices.
- ***n-person Game***: A game played by *n* persons is called an *n-person* game.
- **Zero-sum Game**: A game with two players, where the sum of the payoffs to all players is zero, then it is called a *zero-sum* game. Player's A gain is exactly equal to player B's loss.

5.4 Terminology

- ***Player:*** The participants of the game are called Players
- ***Strategy***: A complete set of plans of action specifying what the player will do under every possible future situation in the game. The strategy can be Pure or Mixed.
- ***Pure Strategy***: A strategy is pure if one knows in advance that this is going to be adopted irrespective of the strategy the other person will adopt.
- ***Mixed Strategy***: The probabilistic combination of the available choices of the strategies is known as mixed strategy.
- ***Payoff***: The profits or returns earned through various strategies followed by each player in relation to the other are known as Payoffs.
- ***Value of Game***: The pay-off at saddle point is called value of the game, where maximin value is equal to minimax value.

5.5 Models of Game Theory:

5.5.1 Two Person Zero Sum Game: A game having two players with gain of one being loss of other is two person zero sum game. Suppose there are two players A and B.A is interested in maximizing his gains while B is interested in minimizing his loss. If both the firms are considering the same three strategies, there will be 9 combinations (3*3=9) available. The payoffs from these strategies are given in the matrix .e.g.

		Player B		
		1	**2**	**3**
Player A	**1**	12	-8	-2
	2	6	7	3
	3	-10	-6	2

The payoffs given in matrix are from A’s point of view, meaning thereby a positive payoff indicates that A has gained and B has lost .The objective is to determine the Best strategy for A and B.

Approaches to solve games:

- **5.5.1.1 The Maximin and Minimax Principle**

This principle is used to select the optimal strategies to be adopted by the two players i.e. the loser and the gainer. Maximin implies that one will choose the maximum out of minimum gains, while Minimax implies that one will choose minimum out of maximum losses. When maximin value is equal to minimax value, the corresponding strategies are known as optimal strategies. The above problem can be solved with this method. If A employs strategy 1,it would expect B to employ strategy 2 as to reduce A’s payoff from strategy 1 to its minimum value i.e. -8 ,representing a loss to A. If A uses strategy 2,it would expect B to use strategy 3.Similarly,if A would be using strategy 3, it would expect B to go for strategy 3.Firm A would like to make the best decision by choosing the maximum out of these minimal pay-off.i.e. (-8, 3,-10). Best is strategy 2 with pay-off 3. This is called Maximin strategy.
On the other hand, When B employs strategy 1, it would expect A to employ strategy 1.If B uses strategy 2, it would expect strategy 2 from A and if B uses strategy 3, it would expect B to employ strategy 2. Later, B would select the strategy that would least benefit its competitor A. Here, B would choose strategy 3 giving pay-off 3 from 12,7,3.Such strategy is known as Minimax strategy. Results show that corresponding to maximin rule of firm A and the minimax rule of fir m B, the pay-off is same.i.e. 3. This is value of game.

- **5.5.1.2 Saddle Point**

A Saddle point is a position in payoff matrix where, the maximum of row minima is equal to the minimum of column maxima. The payoff of the saddle point is called the Value of the game.

Example No. 1: Solve the following game:

	Player B		
		B1	B2
Player A	A1	1	1

	A2	4	-3

Solution:

Player B				
		B1	B2	Row minimum
Player A	A1	1	1	1
	A2	4	-3	-3
Column maximum		4	1	

Maxi (minimum)= V= Max(1,-3)=1

Mini (maximum)=V bar= Min(4,1)=1

Value of the game is 1. The optimal strategy is (A1, B2).

Example No. 2: Solve the game whose pay-off matrix is given below.

	B1	B2	B3	B4	B5
A1	-2	0	0	5	3
A2	3	2	1	2	2
A3	-4	-3	0	-2	6
A4	5	3	-4	2	-6

Solution:

	B1	B2	B3	B4	B5	Row Minima
A1	-2	0	0	5	3	-2
A2	3	2	1	2	2	1
A3	-4	-3	0	-2	6	-4
A4	5	3	-4	2	-6	-6
Column Maxima	5	3	1	5	6	

Maxi (minimum) = V= Max (-2, 1,-4,-6) =1

Mini (maximum = V bar= Min (5, 3, 1, 5, 6)=1

Since V=V bar, there exists a saddle point. Value of the game is 1. The position of the saddle point is the optimal strategy and is given by [A2, B3]

Self-practice Exercise No.1:

Q. No.1 Solve the following game: Player B

Player A

	1	2	3
1	-2	15	-2
2	-5	-6	-4
3	-5	20	-8

5.5.2 Games without Saddle Point

There can be situations when there is no saddle point of a game. Thus pure strategies will not give the solution of the game. One needs to employ mixed strategies. There are various methods to solve such problems:

- **5.5.2.1 Dominance Rule**

In certain situations, it is observed that one of the pure strategies of either player is always inferior to at least one of the remaining. Thus the superior strategies are said to dominate the inferior ones. The superior strategies are said to dominate the inferior ones. In this method, payoff matrix is reduced by deleting the strategies which are dominated by others. The rules of Dominance are as follows:

(i) If all the elements of a row are less than or equal to the corresponding elements of any other, then that row is dominated by the other one.

(ii) If all the elements of a column are greater than or equal to the corresponding elements of any other column, then that column is dominated by the other one.

(iii) Dominated rows and columns may be deleted to reduce the size of the matrix

- **5.5.2.2 Algebraic Method**

The general formula is applicable in this case. Firstly 2*2 matrix is obtained using dominance rule followed by formula.

Suppose the matrix is as below:

	b_1	b_2	Prob.
a_1	a_{11}	a_{12}	p
a_2	a_{21}	a_{22}	1-p
Prob.	q	1-q	

Here, $p = a_{22} – a_{21} / (a_{11} + a_{22}) - (a_{12} + a_{21})$

$q = a_{22} – a_{12} / (a_{11} + a_{22}) - (a_{12} + a_{21})$

Value of game (V) = $a_{11}\, a_{22} – a_{12}\, a_{21} / (a_{11} + a_{22}) - (a_{12} + a_{21})$

Where p = probability of player A using strategy a_1
and q = probability of player B using strategy b_1

Example No. 3: Solve the problem given below by algebraic method

Player B

Player A		**1**	**2**	**3**
	1	3	-1	4
	2	6	7	-2

Solution**:** The first column of the given matrix is dominated by a linear combination of the second and the third columns, as

1/2(-1) + 1/2(4) = 3/2 < 3

1/2(7) + 1/2(-2) = 5/2 < 6

Deleting the first column values, we get the reduced 2x2 matrix which would be used for finding solution to the given problem. Using the usual notations, we have

$p = a_{22} – a_{21} / (a_{11} + a_{22}) - (a_{12} + a_{21})$

$= -2-7 / (-1-2) – (4+7)$

$= 9/14$

$q = a_{22} – a_{12} / (a_{11} + a_{22}) - (a_{12} + a_{21})$

$= -2-4 / (-1-2) – (4+7)$

$= 6/14$

$= 3/7$

$V = a_{11}\, a_{22} – a_{12}\, a_{21} / (a_{11} + a_{22}) - (a_{12} + a_{21})$

= (-1)(-2) – (4x7) / (-1-2) – (4+7)

= 26/14

= 13/7

Accordingly, the optimal policy for A is (9/14, 5/14), for B it is (0,3/7,4/7), and the value of the game is 13/7.

Self-practice Exercise No.2

Q.No.1. Solve the following game:

Player B

Player A		**1**	**2**	**3**
	1	9	8	-7
	2	3	-6	4
	3	6	7	7

Q.No.2. In a game of matching coins with two players, suppose A wins one unit of value when there are two heads, wins nothing when there are two tails and loses ½ unit of value when there one head and one tail. Determine the pay-off matrix, the best strategies for each player and the value of game to A.

Q.No.3. Solve the following game.

Player A	Player B		
	1	7	2
	6	2	7
	5	1	6

5.5.3 2*n and m*2 Games

There can be situations, where either of the players has only two strategies available and the other has n, such game is known as 2*n game or m*2 game. Such problems can be solved using ***Graphical method or Sub Games Method.***

5.5.3.1 Graphic Method: Following steps are followed in this method.

- Construct two vertical axes apart and make a scale on each of them, representing two strategies available to the maximizing player.
- Draw lines to represent each of the minimizing player's strategies.

- Determine the highest point from the lower boundary.
- Formulate (2*2) pay-off sub matrix corresponding to the strategies providing highest point.
- Apply Algebraic method to find the value of the game.

Example No.4: Solve the following game using Graphical approach:

B's Strategy

A's Strategy

	B_1	**B_2**	**B_3**	**B_4**
A_1	8	5	-7	9
A_2	-6	6	4	-2

Solution: Here A has two strategies A_1 and A_2 which, suppose, he plays with probabilities x and 1-x respectively. When B chooses to play b_1, the expected playoff for A shall be 8x+ (-6)(1-x) or 14x-6. Similarly, the expected payoff functions in respect of b_2, b_3 and b_4 can be derived as being 6-x; 4-11x; and 11x-2, respectively. We can represent these by graphically plotting each pay off as a function of x.

GRAPH NO. 1 to be inserted here

The lines are marked b_1, b_2, b_3 and b_4 and they represent the respective strategies. For each value of x, the height of the lines that point denotes the pay-offs of each of B's strategies against (x, 1-x) for A. A is concerned with his least pay-off when he plays a particular strategy, which is represented by the lowest of the four lines at that point, and wishes to choose x so as to maximize this minimum pay-off. This is at K in the figure where the lower envelope (represented by the shaded region), the lowest of the lines at point, is the highest. This point lies at the intersection of the lines representing strategies b_1 and b_3. The distance KL= 0.4 (or -2/5) in the figure represents the game value, V, and x= OL (= 0.4 or 2/5) in the optimal strategy for A.

Alternatively, the game can be written as a 2*2 game as follows:

	B1	B2
A1	8	-7
A2	-6	4

X= [4-(-6)]/[(8+4)-(-7-6)= 2/5
Y= [4-(-7)]/[(8+4)-(-7-6)= 11/25
V= [(8*4)-(-7*-6)]/[(8+4)-(-7-6)]= -2/5
Optimal Strategy for A is (2/5 , 3/5) and for B is (11/25 , 0 , 14/25 , 0).

Example No.5 :Solve the following game:

Strategy		Player B		
		1	**2**	**3**
Player A	**1**	3	-2	4
	2	-1	4	2
	3	2	2	6

Solution: In the given matrix, all the elements in the third column are greater than or equal to the corresponding elements in the first column. Thus, column three is dominated by the first column. So delete column three. The reduced pay-off matrix is as below:

Player B

3	-2
-1	4
2	2

Since no row or column dominates any other row or column, the 3*2 game is to be solved by graphic method. As player B wishes to minimize his maximum loss, the lowest point is to be located on the upper boundary. The expected pay-off equations are then plotted on graph.

GRAPH NO. 2 to be inserted here

The lowest point in the upper boundary is given by the intersection of lines A1 and A2.Thus the solution is reduced to a 2*2 matrix as below:

	B1	B2
A1	3	-2
A2	2	2

X= (2-2)/[3+2-(-2+2)] =0
Y =(2-2)/[3+2-(-2+2)] =4/5
V = [3*2-(-2*2)]/[3+2-(-2+2)]= 2.
Optimal strategies for A (0,0,1) and for B (4/5,1/5)

Self-practice Exercise No.3:

Q. No. 1.Solve the following game graphically:

		Player B		
Strategy		**1**	**2**	**3**
	1	1	3	11
Player A	**2**	8	5	2

Q. No.2. Solve the following game Graphically

		Player B			
Strategy		1	2	3	4
	1	8	5	-7	9
Player A	2	-6	6	4	-2

5.5.3.2 Sub Games Method: In this method, the whole matrix is sub divided in 2*2 matrices. Following steps are used:

- Then each game is solved independently to find the value of game.
- The best sub-game from the point of view of the player having more than two strategies is chosen.
- This provides the optimum solution of the game. As the strategies for this selected sub game will hold good for both the players for the whole game and the value of the so selected sub game will be the value of the complete game.

Example No. 6. Solve the following game using Sub-Games Method:

A	B		
		I	II
	I	2	4
	II	2	3
	III	3	2
	IV	-1	6

Solution:

A	B		
		I	II
	I	2	4
	II	2	3
	III	3	2
	IV	-1	6

The above game does not have saddle point. Hence the player will use mixed strategy. We apply sub games method to solve the above game.

Sub game-1

	I	II
I	2	4
II	2	3

= V = 2

Sub game-II

A	Player B			
		I	II	
	I	2	4	1
	III	3	2	2
		2	1	

V= (2×1+3×2)/1+2 = 8/3

Sub game III

	I	II
I	2	3
IV	-1	6

V = 2

Sub game IV

A	Player B			
		I	II	Odds
	II	2	3	1
	III	3	2	1
	Odds	1	1	2

V= (2×1+3×1)/1+1 = 5/2

Sub game-V

	I	II
I	2	4
IV	-1	6

V = 2

Sub game-VI

A	Player B			
		I	II	
	III	3	2	7
	IV	-1	6	1
		4	4	

V= (3×7+(-1×1))/7+1 = 5/2

Value of sub games

I	II	III	IV	V	VI
2	8/3	2	5/2	2	5/2

Since, A has got more than two choices so Mr. A will select sub game II.

Value of game (v) = 8/3

Probability of selecting strategy

	I	II	III	IV
Player A	1/3	0	2/3	0
Player B	2/3	1/3	-	-

Self-practice Exercise No.4:

Q.No.1.Solve the following game using Sub-Games Method:

Player B

	B1	B2
A1	-6	7
A2	4	-5
A3	-1	-2
A4	-2	5
A5	7	-6

5.5.2 m*n Games

The situations where each of the players has more than two strategies, is known as m*n games. Such problems are formulated and solved through Linear Programming Problem Method (L.P.P.)

5.6 Limitations of Game theory

1. The basic assumption that the players have knowledge of the strategies of opponents is unrealistic.

2. Due to lack of complete information of data, principle of maximin ana minimax cannot be applied.

3. In reality, there are generally more than two players, hence sum of the gains and the losses of the competitor may not be equal to zero.

5.7 Summary

The current chapter deals with the Theory of games. It is concerned with taking the decisions in situations when two or more rational players, having predetermined set of strategies and conflicting interests are involved. Every player tries to use optimal strategy. If there are two players and loss of one is equal to gain of other, the game involved is called two-persons zero sum game having saddle point. The strategies adopted by players in such case are known as pure strategies. In case, no saddle point exists, the players employ mixed strategies. In case of 2*2 game, optimal mix of strategies is obtained by Algebraic method. If the game is of 2*n or m*2 type, it is solved by graphical method. The game with more than two strategies available to each of the players may be reduced using Dominance method if possible and then solved. If it cannot be reduced to 2*2 matrix, it can be formulated as a linear programming problem (L.P.P.) and then solved accordingly.

5.8 Glossary

- **Player*:*** The participants of the game are called Players
- **Strategy**: A complete set of plans of action specifying what the player will do under every possible future situation in the game. The strategy can be Pure or Mixed.
- **Pure Strategy**: A strategy is pure if one knows in advance that this is going to be adopted irrespective of the strategy the other person will adopt.
- **Mixed Strategy**: The probabilistic combination of the available choices of the strategies is known as mixed strategy.
- **Payoff**: The profits or returns earned through various strategies followed by each player in relation to the other are known as Payoffs.
- **Value of Game**: The pay-off at saddle point is called value of the game, where maximin value is equal to minimax value.

5.9 Multiple Choice questions:

1. Two-person zero sum game means that

   (i) Sum of losses to one player equals the sum of gains to other.

(ii) Sum of losses to one player is not equal to the sum of gains to other.

(iii) Both (i) and (ii)

(iv) None of the above.

2. A game is said to be fair, if

   (i) Both upper and lower values of the game are said to be zero.

   (ii) Upper and lower values of the game are not equal.

   (iii) Upper value is more than the lower value of the game.

   (iv) None of the above.

3. The size of the pay-off matrix of a game can be reduced by using the principle of

   (i) Game inversion

   (ii) Rotation reduction

   (iii) Dominance

   (iv) Game transpose

4. A mixed strategy game can be solved by

   (i) Algebraic method

   (ii) Matrix method

   (iii) Graphical method

   (iv) All of the above

5. The pay-off value for which each player in the game always selects the same strategy is called the

   (i) Saddle point

   (ii) Equilibrium point

   (iii) Both (i) and (ii)

   (iv) None of the above

6. Games which involve more than two players are called

   (i) Conflicting games

   (ii) Negotiating games

   (iii) N-person games

   (iv) All of the above

True/False

1. In a two-person game, both the players must have an equal number of strategies.

2. The solution of a two-person game is based on the assumption that the player A will always play his strategy first and the other player, B, would play his strategy thereafter.

3. For a two-person game with, respectively, 6 and 5 strategies available to the two players, a total of 30 conditional pay-offs would be involved.

4. If the largest element in the pay-off matrix is negative, the game should favour player B irrespective of what the optimal strategies of the players are.

5. Mixed strategies for each of the players are determined such that same pay-off would be expected irrespective of the strategy adopted by the opponent.

6. The theory of games is very restrictive in real world applications because of the nature of assumptions made.

5.10 Answer to Check Your Progress

Self- Practice Exercise No.1

Q.1 A1 and B1 or B3, Value of game = -2 for A and 2 for B.

Self- Practice Exercise No.2

Q. No.1 A (1/17, 0, 16/17), B (14/17, 0, 3/17) Value of game= 105/17.

Q. No.2 A (1/4, 3/4), B (1/4, 3/4), Value of game = -1/8

Q. No.3. A (2/5, 3/5, 0), B (1/2, 1/2, 0), Value of game =4.

Self- Practice Exercise No.3

Q. No.1. A (3/11, 8/11): B (0, 2/11, 9/11), Value of game = 49/11

Q. No.2. Value of game = -0.4

Self-practice Exercise No.4:
Q.No.1.Sub –Game VI is the best.
Value of game =23/20
A(0,0,0,13/20,7/20), B(11/20,9/20)

Answers to Multiple Choice questions:

1(i), 2 (i), 3 (iii), 4 (iv), 5 (iii), 6 (iii)

Answers to True/False questions

1 T, 2 F, 3 T, 4 T, 5 T, 6 T.

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5.13 Model Questions:

1. Consider the following parlor game to be played between two players. Each player begins with three chips: one red, one white and one blue. Each chip can be used only once.
   To begin, each player selects one of her chips and places it on the table, concealed. Both players then uncover the chips and determine the payoff to the winning player. In particular, if both players play the same kind of chip, it is a draw; otherwise, the following table indicates the winner and how much she receives from the other player. Next, each player selects one of her two remaining chips and repeats the procedure, resulting in another payoff according to the following table. Finally, each player plays her one remaining chip, resulting in the third and final payoff.

Winning Chip	**Payoff ($)**
Red beats White	50
White beats Blue	40
Blue beats Red	30
Matching Colors	0

Formulate this problem as a two-person zero-sum game by identifying the form of the strategies and payoffs.

2. Two companies share the bulk of the market for a particular kind of product. Each is now planning its new marketing plans for the next year in an attempt to wrest some sales away from the other company. (total sales for the product are relatively fixed, so one company can increase its sales only by winning them away from the other.) Each company is considering three possibilities:

(1) Better packaging of the product

(2) Increased advertising

(3) A slight reduction in price.

The costs of the three alternatives are quite comparable and sufficiently large that each company will select just one. The estimated effect of each combination of alternatives on the increased percentage of the sales for company 1 is as follows:

Strategy		**Player 2**		
		1	**2**	**3**
Player 1	**1**	2	3	1
	2	1	4	0
	3	3	-2	-1

Each company must make its selection before learning the decision of the outer company.

(a) Without eliminating dominated strategies, use the minimax (or maximin) criterion to determine the best strategy for each company.

(b) Now identify and eliminate dominated strategies as far as possible. Make a list of the dominated strategies, showing the order in which you were able to eliminate them. Then show the resulting reduced payoff table with no remaining dominated strategies.

3. Two competing firms must open their next branch at one of the three cities: A, B and C, whose distance profile is given here. If both companies open their branches in the same city, they will split the business evenly. If, however, they build the branches in different cities, the company that is closer to a given city will get that entire city's business. If all the three cities have the same amount of business, where should the firm open the branches?

Distance Profile	
From A to B	25 km
From A to C	18 km
From B to C	22 M

4. The management of a corporation is in the process of deciding whether to agree to negotiate with the striking union now or delay. The decision is difficult because the management does not know the union leadership's position. The union leaders may be adamant and insist on their original demands, they may be ready to compromise or they may be ready to yield and accept the original management offer. The matrix of payoffs to management, as management sees it, is (in Rs. 1 million units).

	Union Position		
	B_1 Adamant	**B_2 Compromise**	**B_3 Yield**
A_1 Negotiate	-2	1	2
A_2 Delay	5	-2	-3

(a) Solve the management's problem.

(b) What should be the union's strategy?

(c) Discuss the implications of a conclusion to adopt a random strategy.

5. Solve the following game:

Player A	**Player B**	
	B_1	**B_2**
A_1	3	4
A_2	-3	12
A_3	6	-2
A_4	-4	-9
A_5	5	-3

6. Determine the Optimal minimax strategies for each player in the following game.

	B_1	B_2	B_3	B_4
A_1	-5	2	0	7
A_2	5	6	4	8
A_3	4	0	2	-3

7. Solve the following game:

Player A	**Player B**					
		I	**II**	**III**	**IV**	**V**	**VI**
	1	4	2	0	2	1	1
	2	4	3	1	3	2	2
	3	4	3	7	-5	1	2
	4	4	3	4	-1	2	2
	5	4	3	3	-2	2	2

8. Use Dominance principle to simplify the rectangular game with the following pay-off matrix and then solve graphically.

Player A	**Player B**				
		I	**II**	**III**	**IV**
	1	17	5	8	5
	2	7	3	15	8
	3	13	6	16	4
	4	8	7	14	3

8. "The two-person, zero-sum game is unrealistic." Elucidate the statement bringing out the limitations of game theory, if any.

9. A game refers to a situation of business conflict." Comment on the situation.

10. Show how a two person zero sum game problem can be reduced to a linear programming problem.

11. By using Dominance property, solve the following game:

Player B				
	1	**2**	**3**	**4**

Player A	**1**	1	7	2
	2	6	2	7
	3	6	1	6

12. Use Graphical method in solving the following game:

Player B					
		1	**2**	**3**	**4**
Player A	**1**	2	2	3	-2
	2	4	3	2	6

13. Suppose the two firms are competing for a market share of the sales for a particular product. Each firm is considering what promotional strategy to employ for the coming sales period. Assume that the following payoff matrix describes the increase in market share for firm A and the decrease in market share for Firm B. Determine the Optimal strategies for each firm.

Firm B

Firm A	**No Promotion**	**Moderate Promotion**	**Extensive Promotion**
No Promotion	5	0	-10
Moderate Promotion	10	6	2
Extensive Promotion	20	15	10

14. Solve the following game using maximin (minimax) principle. Determine the pure strategies for each player and value of game.

Player B

	B1	**B2**	**B3**	**B4**	**B5**	**B6**
A1	1	1	2	4	5	-5
A2	3	-3	4	3	2	4
A3	6	2	3	5	7	5
A4	2	1	3	4	6	0

Chapter 6

Sequencing Problems

Structure

6.1 Objectives

After reading the following sections the reader will be able:

- To understand the meaning of the Sequencing Problems

- To understand various notations and terms relating to Sequencing
- To understand the assumptions of Sequencing Theory
- To know the methods of solving various types of Sequencing problems

6.2 Introduction

Sequencing problems are helpful in situations when there is a choice in the order in which various activities can be performed .e.g. Machines need repairs, Jobs processing in manufacturing, aircrafts waiting for landing, customers waiting for service etc. More commonly, it is applicable in case of the jobs to be performed in a factory. Sequencing problem arises in the situation when a set of either two or more jobs are to be performed and a set of one or more facilities are available to perform those jobs. Such problems are also known as scheduling problems. The objective is to reduce idle time by making effective use of resources and get higher productivity.

In developing a schedule, the resources required are to be matched with the available resources like machine hours, labour hours etc. Thus determination of an effective and efficient order for processing various jobs on machines is to be determined first. This is known as an Optimum order (sequence) of performing a number of jobs by number of facilities available.

6.2.1 Definition

Suppose there are n jobs (1,2,3,…,n), each of which has to be processes done at a time at each of m machines A , B, C, …The order of processing each job through machines is given(e.g. job 1 is processed through machines A, C,B)in this order. The time that each job must require on each machine is known. The problem is to find a sequence among $(n!)^m$ number of all possible sequences(or combinations) for processing the jobs so that the total elapsed time for all the jobs will be minimum.

Mathematically;

Let A_i = Time for job_i on machine A

B_i = Time for job_i on machine B, etc.

T = Time from start of first job to completion of the last job.

Then the problem is to determine for each machine a sequence of jobs i_1, i_2, i_3,… ,i_n, where (i_1, i_2, i_3,… i_n) is the permutation of the integers which will minimize T

6.2.2 Notations:

t_{ij} =Processing time or time required for job *i* through machine *j*.

T = Total elapsed time for processing all the jobs including idle time, if any.

I_{ij} = Idle time on machine *j* from the end of (i-1)th job to the start of job *i*.

6.2.3 Terminology:

Number of Machines**:** It refers to the number of service facilities through which a job must pass before it is completed. For example, a book to be published has to be processed through composing, printing, binding, etc. In this case, the book constitutes the job and the various processes constitute the number of machines.

Jobs: The jobs or customers are the units that stimulate sequencing.

Processing Order**:** It refers to the sequence (order) in which various machines are required for completing the job.

Processing Time**:** It indicates the time required by a job on each machine.

Idle Time on a Machine**:** It is the time for which a machine does not have a job to process, i.e. idle time from the end of job (i-1) to the start of jobs *i*.

Total Elapsed Time**:** It is the time between the starting of the first job and completion of the last one.

No Passing Rule**:** It implies that passing is not allowed, i.e. the same order of the jobs is maintained over each machine. For example, if *n* jobs are to be processed on two machines, *A* and *B* in the order *AB*, then each job must go to Machine *A* first and then to Machine *B*.

Measure of Efficiency**:** Measure of Efficiency is the optimization of goal.

Arrival Pattern: The way in which various jobs arrive at service facility is known as arrival pattern. If all the jobs arrive simultaneously then it is called a static pattern. Continuous arrival of jobs is known as dynamic pattern.

6.2.4 Assumptions

- No machine can process more than one operation at a time.
- Every operation once started must be performed till completion
- Each operation must be completed before starting any other operation.
- Time intervals for processing are independent of the order in which operations are performed.
- There is only one of each type of machine.
- A job is processed as soon as possible subject to ordering requirements

- All jobs are known and are ready to start processing before the period under consideration begins
- The time required to transfer jobs between machines is negligible

6.3 Methods of Solving Sequencing Problems

There are two methods of solving sequencing problems.

- Gantt Charts
- Algorithm Method

6.3.1 Gantt Charts: Simple problems of sequencing involving two machines only can be solved through this method.

Example No.1.Suppose there are two jobs J1 and J2, each needs to be processed on two machines M1 and M2. What order of performance of the jobs will involve the least time? The required processing times are as follows:

Processing Time (Hours)

	Machine	
Job	M1	M2
J1	4	10
J2	8	7

Solution (INSERT Graph No.1 here)

6.3.2 Algorithm Method: There are various methods of solving different kinds of sequencing problems as discussed in the following sections:

6.3.2.1 Processing n jobs through Two Machines (Johnson's Algorithm)

Suppose two machines A and B are available for scheduling of many jobs. For solving such problem, following steps are involved:

1. Select the least processing time. In case of tie, either of the smallest processing time should be selected.

2. If the least processing time is associated with Machine A, assign the job as the first job. In case, it is associated with Machine B, that job will be scheduled the last. Eliminate this scheduled job from further consideration.

3. Locate next lowest processing time and schedule the jobs accordingly.

4. Repeat this process till all the jobs have been ordered. The resulting ordering will minimize the elapsed time T.

Example No.2. A book binder has one printing press, one binding machine and the manuscripts of a number of different books. The time required for performing the printing and binding operations for each book are shown as follows:

Books	1	2	3	4	5	6
Printing time (hrs)	30	120	50	20	90	100
Binding time (hrs)	80	100	90	60	30	10

Determine the order in which the books should be processed in order to minimize the total time required to turn out all the books.

Solution:

Using Bellman Johnson’s technique, we schedule as follows:

Step1. Selecting the least processing time for printing or binding for all the books, we find the figure as 10.

Step2. Since figure of 10 belongs to Binding operation for Book 6, this book is to be scheduled as last and book 6 not to be considered for further scheduling.

			6

Step3. Selecting next best time, it is 20 hrs which is associated to printing for book 4. Hence book 4 is scheduled as first. The scheduling so far achieved is as follows:

4					6

Step 4. Repeating step 3 for other unscheduled books, the final schedule works out like this

4	1	3	2	5	6

The table for this scheduling can be drawn as given below:

0 20 50 100 220 310 400

Printing	4 (20)	1 (30)	3 (50)	2 (120)	5 (90)	6 (100)	
Binding		4 (60)	1 (80)	3 (90)	2 (100)	5 (30)	6 (10)

0 20 80 160 250 350 380 410 420

Self-Practice Exercise No.1:

Q.No.1.There are seven jobs each of which has to go through the machine M_1 and M_2 in the order M_1 M_2. Processing times in hours are given as:

Job	1	2	3	4	5	6	7
Machine M_1	3	12	15	6	10	11	9
Machine M_2	8	10	10	6	12	1	3

Determine a sequence of these jobs that will minimize the total elapsed time.

Q.No.2. A company has 6 jobs in hand coded 'A' to 'F'. All the jobs have to go through two machines 'MI' and 'MII'. The time required for each job on each machine, in hours, is given below:

	A	B	C	D	E	F
MI	3	12	18	9	15	6
MII	9	18	24	24	3	15

Draw a sequencing table scheduling the six jobs on the two machines.

6.3.2.2 Processing n jobs through Three Machines

In this case Johnson's rule is applicable with a little modification. Consider *n* jobs (1,2,3…,n) processing on three machines A,B,C in the order ABC. The optimal sequence can be obtained by converting the problem into a Two-machine problem by using the following steps:

1. Find the minimum processing time for the jobs on the first and last machine and maximum processing time for the second machine.
2. Check the following condition, i.e.,

$$\text{Minimum of } a_i \geq \text{Maximum of } b_i$$

Or

$$\text{Minimum of } c_i \geq \text{Maximum of } b_i$$

3. If none of the conditions given in step 2 are satisfied, the method cannot be applied.
4. If at least one of the conditions is satisfied, we define two machines M_1 & M_2 such that the processing time on M_1 & M_2 are given by

M_1 (new machine 1) is equal to $a_i + b_i$, where i = 1,2,3, …,n, and;
M_1 (new machine 2) is equal to $b_i + c_i$, where i = 1,2,3, …,n.

5. Further new machines M_1 & M_2, optimal sequence is obtained using two-machine algorithm.

Example No. 3: A garment manufacturer has to process 5 items through three stages of production, viz. cutting, sewing, and pressing. The time taken for each of these items at the different stages is shown in hours through the following table:

Items Jobs	Cutting A_i	Sewing B_i	Pressing C_i
1	40	50	80
2	90	60	100
3	80	20	60
4	60	30	70
5	50	40	110

Find an order in which these items are to be produced through these stages so as to minimize the total time involved.

Solution:

(i) Min $A_i \geq$ max B_i

(ii) Max $C_i \geq$ max B_i

Is satisfied, therefore, the given problem can be converted to a problem of 5 items and 2 stages.

Thus, define

$G_i = A_i$(Cutting)+ B_i(Sewing), i = 1,2, …, 5

And $H_i = B_i$(Sewing) + C_i(Pressing), i = 1,2, …,5

	G_i	H_i	Optimum sequence is then any of the following				
1	90	130	1	4	5	2	3
2	150	160	1	5	4	2	3
3	100	80	5	1	4	2	3
4	90	100	5	4	1	2	3
5	90	150	4	1	5	2	3
			4	5	1	2	3

Total completion time is shown in the table below.

Sequence	Machine A	Machine B	Machine C	Idle time of B	Idle time of C

	Time in	Time out	Time in	Time out	Time in	Time out	Time in	Time out
1	00	40	40	90	90	170	40	90
4	40	100	100	130	170	240	10	00
5	100	150	150	190	240	350	20	00
2	150	240	240	300	350	450	50	00
3	240	320	320	340	450	510	20	00

Total elapsed time = 510
Total idle time of Machine B (sewing) = 140 hours
Total idle time of Machine C (pressing) = 90 hours.

Self-Practice Exercise No.2:

Q.No.1.Find the sequence that minimizes the total elapsed time required to complete the following tasks.

Tasks	A	B	C	D	E	F	G
Time on 1^{st} machine	3	8	7	4	9	8	7
Time on 2^{nd} machine	4	3	2	5	1	4	3
Time on 3^{rd} machine	6	7	5	11	5	6	12

Q.No.2. Find the sequence for the following eight jobs that will minimize the total elapsed time for the completion of all the jobs. Each job is processed in the same order.

Time for machines	JOBS							
	1	2	3	4	5	6	7	8
A	4	6	7	4	5	3	6	2
B	8	10	7	8	11	8	9	13
C	5	6	2	3	4	9	15	11

The entries give the time in hours on the machine.

6.3.2.3 Processing n jobs through *k* Machines

Consider *n* jobs (1,2,3…,n) processing on *k* machines M_1, M_2, …, M_k in the same order. The following steps are to be used to obtain an optimal sequence:

1. Find minimum M_i and M_{ik} and maximum of each of M_{i2}, M_{i3}, …, M_{ik-1} for i = 1,2,…, n.

2. Check the following conditions, i.e.,

$$\text{Minimum of } M_{i1} \geq \text{Maximum of } M_{ij}\text{ ; for } j = 2,3, \ldots , k-1$$

Or

$$\text{Minimum of } M_{ik} \geq \text{Maximum of } M_{ij}\text{ ; for } j = 2,3, \ldots , k-1$$

3. If the conditions above are not satisfied, the method cannot be applied.
4. In addition to step 2, if $M_{i2} + M_{i3} + \ldots + M_{ik-1} = C$, where C = positive fixed constant for all, $i = 1,2, \ldots ,n$. determine the optimal sequence for n jobs, where the two machines (new) are M_l and M_k .
5. If the condition given in step 4 is not satisfied, we define two machines G and H such that $G_i = M_{i1} + M_{i2} + \ldots + M_{ik-1}$ and;

   $H_i = M_{i2} + M_{i3} + \ldots + M_{ik}$; where $i = 1,2, \ldots ,n$.

   Now the optimal sequence of performance of jobs on machines G and H is obtained by using algorithms`` of two machines.

Example No. 4 : Find the optimal sequence for processing 4 jobs 1, 2, 3, 4 on four Machines A, B, C, D in the order ABCD. Processing times are given in the table below.

Jobs	Processing times (in hours)			
	Machines			
	A	B	C	D
1	15	5	4	14
2	12	2	10	12
3	13	3	6	15
4	16	0	3	19

Solution:

From the given example table one get that for initial and last machines,

Min A_i = min processing time on first machine = 12

Min D_i = min processing time on last machine = 12

For intermediate Machines,

Max B_i = max processing time on last machine = 5

Max C_i = max processing time on last machine = 10

Since, min A_i> max B_i and max C_i (both), the problem can be reduced to a problem involving only two machines G and H with processing times as shown in the next table.

Jobs	Processing Times

	Machines	
	G_i	H_i
1	15+5+4= 24	5+4+14 = 23
2	12+2+10 = 24	2+10+12 = 24
3	13+3+6 = 22	3+6+15 = 24
4	16+0+3 = 19	0+3+19 = 22

Using the algorithm for solving a sequencing problem of n jobs and 2 machines, one get the optimal sequence as:

4	3	2	1

Total elapsed time can be calculated as shown in the table below.

	Machines							
	A		B		C		D	
	Time in	Time out	Time in	Time out	Time in	Time out	Time in	Time out
4	0	16	16	16	16	19	19	38
3	16	29	29	32	32	38	38	53
2	29	41	41	43	43	53	53	65
1	41	56	56	61	61	65	65	79

Therefore, total elapsed time = 79 hours

Example No. 5

Determine the optimal sequence of performing 5 jobs on 4 machines. The machining of each job is required in the order ABCD and the process timings are as follows:

Jobs	Machine			
	A	B	C	D
1	8	3	4	7
2	9	2	5	5
3	6	4	5	8
4	12	5	1	9
5	7	1	2	3

Here Min A_i = 6, Max B_i = 5, Max C_i = 5and Min D_i = 3
Also since Min A_i>Max B_i, Max C_i the first of the conditions is satisfied. Thus, we can apply the algorithm. First, we prepare the consolidation table as given in Table 8.5

Consolidation Table

Job	$X_i = A_i + B_i + C_i$	$Y_i = B_i + C_i + D_i$
1	15	14
2	16	12
3	15	17
4	18	15
5	10	6

According to the information in the consolidation table, the optimal sequence of jobs would be as follows:

3 4 1 2 5

This sequence would involve a total completion time of 50 hours, as shown in the Table given below:

Calculation of Total Elapsed Time (T)

Jobs	Machine A		Machine B		Machine C		Machine D	
	In	Out	In	Out	In	Out	In	Out
3	0	6	6	10	10	15	15	23
4	6	18	18	23	23	24	24	33
1	18	26	26	29	29	33	33	40
2	26	35	35	37	37	42	42	47
5	35	42	42	43	43	45	47	50

Therefore, total elapsed time = 50 hours

Self-Practice Exercise No. 3.

Q.No.1.We have four jobs, each of which has to go through the machines M_j, j= 1,2…6 in the order $M_1, M_2 \ldots M_6$.

Processing time (in hours) is given below.

	M_1	M_2	M_3	M_4	M_5	M_6
Job A	18	8	7	2	10	25
Job B	17	6	9	6	8	19
Job C	11	5	8	5	7	15
Job D	20	4	3	4	8	12

Determine the sequence of these four jobs that minimizes the total elapsed time.

6.3.2.4 Processing 2 jobs through *k* Machines

In case of two jobs i.e. J_1 and J_2 and k machines i.e. A, B, ... , k, the processing times are known to be $A_1, A_2, B_1, B_2, \ldots, k_1, k_2$. The objective is to find the optimal order of jobs to be processed that will minimize the total elapsed time. It is assumed that each machine can work only one job at a time. Moreover, alternate sequencing of machines is not permissible. The problems of such kind are solved through graphic method.

1. In this method, jobs J_1 and J_2 are represented on X-axis and Y- axis respectively. On these two axes, the processing time on different machines is marked in given order. The horizontal line in the graph will represent the work on job 1only and job 2 remains idle. Similarly, a vertical line in the graph represents work on job 2 while job 1 remains idle. A line at 45^0angle will represent the simultaneous work on both the jobs. It is pertinent to mention here that a job will be processed on a machine if the machine is idle. A horizontal or vertical line will occur when some job is idle but the machine which is the next to this job is not idle. Moreover, both the jobs cannot be processed on the same machine simultaneously. Later the shortest line, composed of horizontal line, vertical line and 45^0 line, from the origin to the end would represent the optimal sequence. It is the best path which also minimizes the idle time of jobs J_1 and J_2

Example No.6: Using graphical method, calculate the minimum time needed to process Job 1 and Job 2 on five machines A, B, C, D and E i.e. for each machine find the job which should be done first. Also, calculate the total time needed to complete both jobs.

		Machine				
Job 1	Sequence:	A	B	C	D	E
	Time (hours):	1	2	3	5	1
Job 2	Sequence:	C	A	D	E	B
	Time (hours):	3	4	2	1	5

Solution: Graph No. 2 is to be inserted here

Total Elapsed Time:

Job 1 12+3 = 15hours

Job 2 15+Nil =15hours

Thus it is 15 hours

Idle Time:

Job 1 = 3 hours

Job 2 = Nil

Self-Practice Exercise No. 4

Q.No.1.Two jobs are to be processed on four machines A, B, C , D. the technological order for these machines is as follows.

Job 1	A	B	C	D
Job 2	D	B	A	C

Processing periods (time) are given in the following table.

	Machines			
	A	B	C	D
Job 1	4	6	7	3
Job 2	4	7	5	8

Find the optimal sequence of jobs on each of the machines.

Q.No.2. A machine shop has four machines A, B, C, D. Two jobs must be processed through each of these machines. The time (in hours) taken on each of the machines and the necessary sequence of jobs through the shop are given below:

Job 1	Sequence	A	B	C	D
	Time	2	4	5	1
Job 2	Sequence	D	B	A	C
	Time	6	4	2	3

Use graphic method to obtain minimum elapsed time.

6.4 Summary

Sequencing means determining the optimal sequence or order to perform given jobs. Since, the number of jobs as well as the number of machines on which the jobs are to be performed varies, the problem arises. There are different methods available to solve sequencing problems. When there are n jobs to be performed on two machines one by one, the optimal sequence is the one which leads to minimum total time as well as minimum idle time. Minimum processing time from all the jobs is determined and the job is to be performed on the machine with minimum processing time first. When optimal order is obtained, total elapsed time T is calculated. When there are n jobs to be performed on three machines, certain conditions need to be satisfied. i.e. Min $A_i \geq$Max B_i and/or Min $C_i \geq$Max B_i. In such cases, the problem is converted into a two machine problem. T is calculated according to optimal order time. Similarly, in case machines are more than three, again some conditions need to be satisfied and then problem converted into two machines to be solved to get an optimal sequence. There can be a case of two jobs to be processed on more than two machines in different orders. Such kinds of problems are solved with the help of graphical method.

6.5 Glossary

Jobs: The jobs or customers are the units that stimulate sequencing.

Processing Order: It refers to the sequence (order) in which various machines are required for completing the job.

Processing Time: It indicates the time required by a job on each machine.

Idle Time on a Machine: It is the time for which a machine does not have a job to process, i.e. idle time from the end of job (i-1) to the start of jobs *i*.

Total Elapsed Time: It is the time between the starting of the first job and completion of the last one.

No Passing Rule: It implies that passing is not allowed, i.e. the same order of the jobs is maintained over each machine. For example, if *n* jobs are to be processed on two machines, *A* and *B* in the order *AB,* then each job must go to Machine *A* first and then to Machine *B.*

Measure of Efficiency: Measure of Efficiency is the optimization of goal.

Arrival Pattern: The way in which various jobs arrive at service facility is known as arrival pattern. If all the jobs arrive simultaneously then it is called a static pattern. Continuous arrival of jobs is known as dynamic pattern.

6.6 True or False (Questions)

1. In sequencing problems, the effectiveness is a function of the order in which the given tasks are performed.
2. In sequencing problems involving n jobs and 3 machines, the algorithm is based on the assumption that each of the jobs requires processing on three machines in the same order.
3. Gantt chart provides a very effective technique for solving large-sized sequencing problems.
4. Total elapsed time is determined by the point of time at which the first of the n jobs goes to machine A until the point on which the last job comes off machine B.
5. The principle used for processing optimally certain jobs on two machines can be summarized as follows:

   Job j precedes job j+1 if min $(A_j, B_{j+1}) < \min(A_{j+1}, B_j)$
6. There is no general solution available for solving sequencing problems involving processing of n jobs through 3 machines.
7. If Min $A_i \geq$ Max B_i or Min $C_i \geq$ Max B_i or both, then the problem is replaced by an equivalent problem involving n jobs and 2 fictitious machines G and H, where $G_i = A_i + B_i$ And $H_i = B_i + C_i$.

8. If some jobs are processed on K machines in the order A, B, C …. , K, then a standard method can be employed when Min ≥ Max B_i ≥ Max C_i ≥ … ≥ Max J_i and/or Min K_i ≥ Max B_i ≥ Max C_i ≥ … ≥ Max J_i.
9. Sequencing problems involving processing f two jobson k machines can be solved graphically.
10. A 45 degree line on graph, where two jobs are shown on the two axes, indicates simultaneous work on the jobs.

6.7 Answer to Check Your Progress:

Self-Practice Exercise No.1

Ans.No.1. 1-4-5-3-2-7-6, Total time = 67 hours.
Ans.No.2. Optimum sequence: A-F-D-B-C-E, Min elapsed time = 96 hours.
Idle time: 33 for MI and 3 hours for MII.

Self-Practice Exercise No.2

Ans.No.1. A-D-G-F-B-C-E or A-D-G-B-F-C-E
Ans.No.2. 4-5-3-1-2-8-7-6., Min time is 81 hours.

Self-Practice Exercise No.3

Ans.No.1. C-A-B-D

Self-Practice Exercise No.4

Ans.No.1. Total elapsed time = 24 hours.
Ans.No.2.The total elapsed time is 15 hours.

Ans. of True/False (questions)

1. T
2. T
3. F
4. T
5. T
6. T
7. T
8. T
9. T

10. T

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6.10 Terminal and Model Questions

1. What do you mean by "no passing" rule in sequencing?

2. Explain the Johnson's method of determining an optimal sequence for processing n jobs on two machines giving a suitable example.

3. "Simple sequencing problems often solved by the use of the ' Gantt Chart "; do you agree? Give reasons for your answer.

4. Elucidate how to process n jobs through m machines.

5. Explain the various sequencing models that are available for solving the sequencing problems. Give suitable examples.

6. Determine the sequence for the jobs that will minimize the total elapsed time. Provide the optimal job sequence involving three machines M1, M2, M3 for the following.

		Jobs				
		J_1	J_2	J_3	J_4	J_5
Time	M1	7	12	11	9	8
On	M2	8	9	5	6	7
Machines	M3	11	13	9	10	14

7. The company has 6 jobs which go through 3 machines X, Y, Z in the order XYZ. The processing time in minutes for each job on each machine is as follows:

Machine	Jobs					
	1	2	3	4	5	6
X	18	12	29	35	43	37
Y	7	12	11	2	6	12
Z	19	12	23	47	28	36

8. We have five jobs, each of which has to go through machines A and B in the order AB. Processing times are given in Table below:

Processing-times in hours

Jobs	Time	
	Machine(A_i)	Machine(B_i)
1	5	2
2	1	6
3	9	7
4	3	8
5	10	4

Determine a sequence of these jobs that will minimize the total elapsed time, T.

9. Determine the optimal sequence of 5 jobs on three machines which minimizes the total elapsed time based on the information given in Table below:

Processing time on the machines A, B, C

Job	A_i	B_i	C_i
1	3	3	5
2	8	4	8
3	7	2	10
4	5	1	7
5	2	5	6

10. Using graphical method, calculate the minimum time needed to process Job 1 and Job 2 on five machines A, B, C, D and E, i.e. for each machine find the job which should be done first. Also, calculate the total time needed to complete both jobs.

Job1	Sequence:	A	B	C	D	E
	Time (hours):	4	6	8	10	12
Job2	Sequence:	C	A	D	E	B
	Time (hours):	6	4	10	6	4

LESSON- 7

INVENTORY MODELS

STRUCTURE

7.0 OBJECTIVES

This chapter will help the students to

- understand the concept of Economic order quantity
- comprehend EOQ through Algebraic Method, Tabular Method and Graphic Method
- make a decision when a choice has to be made between EOQ and quantity discount
- explain purchasing inventory model with one price break and multiple price breaks

7.1 INTRODUCTION

Today is the era of industrialization which poses a big problem for the management to reduce the cost and at the same time to control the cost. Out of the total cost, nearly more than 50% is the cost of material. Thus an effective control over the inventory is a major concern in order to boost up concern's profit. Purchasing department has to be very efficient to make a decision for a quantity of material to be purchased to ensure maximum economy in purchasing any item of material. Along with the purchasing cost, purchasing department has also to focus on ordering cost and carrying cost. Ordering cost is associated with placing an order; follow up of order, transportation, receiving of goods and inspection cost etc. However carrying cost is related with holding and storing inventories in the stores. If the inventory is purchased in large quantity, ordering cost will reduce but at the same time, the storage cost will get increased. On the other hand if inventory is purchased in small quantity in each order, the ordering cost will get increased but the carrying cost will get reduced. So, there is the need of proper inventory models which can keep a check on ordering cost and carrying cost and at the same time will bring economy in purchasing. Economic order quantity is the most popular technique which keeps equilibrium in ordering cost and carrying cost and keeps total cost at minimum.

9.2 ECONOMIC ORDER QUANTITY

Economic Order Quantity (EOQ) is one of the oldest and widely used inventory control techniques. It is also known as optimum or standard order quantity. The credit for developing the concept of economic order quantity goes to Ford W. Harries who used this technique in one of his publication in the year 1915 and is still used by a large number of organizations.

EOQ technique helps in determining the most economical purchase order quantity or lot size where total cost is minimum and annual ordering cost for a material is exactly equal to annual carrying cost.

ASSUMPTIONS

1) Demand for the item is known and constant.
2) The lead time (the time between the placement of the order and the receipt of the order) is known and constant.
3) The receipt of inventory is instantaneous or the inventory from an order arrives in one batch at one point of time.
4) Quantity discounts are not to be availed.

5) The variable costs related with inventory acquisition are ordinary cost (cost of placing an order) and carrying cost (cost of holding or storing inventory over time).
6) If orders are placed at the right time, there will not be any stock out position.

DETERMINATION OF ECONOMIC ORDER QUANTITY (E.O.Q)

Following three methods are available to determine E.O.Q
(a) Algebric Method
(b) Tabular Method
(c) Graphic Method

A) Algebric Method :- As already discussed, optimal order quantity is the point where total cost is minimum and the ordering cost is equal to the carrying cost. The same concept is followed to determine economic order quantity by algebric method.

The following variables are used to calculate ordering cost, carrying cost and economic order quantity.
D = Annual demand in units
Co = Ordering cost per order
Cc = Carrying cost per unit per year
Q = Number of units per order
Q* = Economic order quantity or optimum number of units per order

The steps followed to determine E.O.Q are as follows:-

1st Step
Annual ordering cost = Number of orders placed in a year X Ordering cost per Order

$$= \frac{\text{Annual Demand}}{\text{Number of Units in each order}} \times \text{Ordering cost per order}$$

= (D/Q) X (Co) or D Co/Q

2nd Step
Annual holding or carrying cost = Average Inventory X Carrying cost per unit Per year

Or (Order Quantity /2) X Carrying cost per unit per year

Or (Q/2) X (Cc) = Q Cc/2

3rd Step

Optimal order quantity is found when ordering cost is equal to carrying cost thus

(D/Q) X Co = Q Cc/2

$Q^2 = 2\ D\ Co\ /Cc$

EOQ = Q* = $\sqrt{2DCo/Cc}$

The equation thus derived for economic order quantity can be used directly to solve inventory problems.

Similarly under, EOQ technique total variable cost is calculated by using the following formula.

Total Variable cost = $\sqrt{2\ D\ Co\ Cc}$

Example 1

PQ ltd purchases 5000 parts of machinery for its annual requirements. Each part costs ₹ 30. The ordering cost per order is ₹ 15 and the carrying cost is 10% of the average inventory per year. Find out E.O.Q

Solution:-	Annual Demand (D)	=	5000 units
	Unit cost	=	₹ 30
	Ordering cost (Co)	=	₹ 15
	Carrying Cost (Cc)	=	10% or (30X10)/100
	Therefore Cc	=	₹ 3
	EOQ	=	$\sqrt{2DCo/Cc}$
		=	$\sqrt{\frac{2 X 5000 X 15}{3}}$
	EOQ	=	224 units

Example 2

A manufacturer buys certain equipment at ₹ 30 per unit. The annual requirement for such equipment is 1000 units. Further data available is as under:

Annual return of investment = 5%

Rent, Insurances, taxes per unit per year = ₹ 2.5

Cost of placing an order = ₹ 50

Determine the economic order quantity.

Solution:- Annual requirement (D) = 1000 units

Unit cost = ₹ 30 per unit

Ordering cost per order (Co)= ₹ 50

Carrying Cost per unit (Cc) = (30X5%) + 2.5

Therefore Cc = 1.5 +2.5 = ₹ 4

EOQ = $\sqrt{2DCo/Cc}$

= $\sqrt{\frac{2 X 1000 X 50}{4}}$

EOQ = 158 units

Example 3

Annual Consumption = 20,000 units

Ordering and receiving cost = ₹ 20 per order

Annual Carrying cost = 10% of inventory value

Cost of material per unit = ₹ 50

Find out Economic order quantity (i) in units (ii) in Rupees

Solution:- (i) Calculation of E.O.Q (in Units)

Annual consumption (D) = 20000 units

Unit cost = ₹ 50

Ordering cost (Co)	=	₹ 20 per order
Carrying Cost (Cc)	=	(50X10)/100
Therefore Cc	=	₹ 5
EOQ	=	$\sqrt{2DCo/Cc}$
	=	$\sqrt{\frac{2 X 2000 X 20}{5}}$
EOQ	=	400 units

(ii) Calculation of E.O.Q (in Rupees)

Annual consumption (in ₹)(D)	=	20000 X 50
	=	₹ 10,00.000
Ordering cost (Co)	=	₹ 20 per order
Carrying Cost (Cc)	=	10% of inventory cost
EOQ	=	$\sqrt{2DCo/Cc}$
	=	$\sqrt{\frac{2 X 1000000 X 20}{10\%}}$
EOQ	=	₹ 20,000

Note :- When E.O.Q is to be calculated in Rupees then annual Demand (D) is to be taken in Rupees and inventory cost (Cc) is taken in percentage form instead of inventory cost per unit per year.

Example 4

The annual demand for an item is 6400 units. The unit cost is ₹ 12 and inventory carrying charges 25% per annum. If the cost of one procurement is ₹ 150. Determine

(i) EOQ
(ii) Number of orders per year
(iii) Time between two consecutive orders
(iv) Optimal Cost

Solution:- Annual requirement (D) = 6400 units

Unit cost = ₹ 12

Ordering cost per order (Co)= ₹ 150

Carrying Cost (Cc) = (12X25%) = ₹ 3

(i) EOQ = $\sqrt{2DCo/Cc}$

$$= \sqrt{\frac{2 X 6400 X 150}{3}}$$

EOQ = 800 units

(ii) Number of orders per year = $\frac{\text{Annual Demand (D)}}{\text{EOQ}}$

= 6400/800

= 8 orders

(iii) Time between two consecutive orders = (12/8) months

= 1.5 months

(iv) Optimal Cost = Material Cost + Total Variable cost

Material cost = 6400 X 12 = ₹ 76800

Total Variable Cost = $\sqrt{2\ D\ Co\ Cc}$

= $\sqrt{2 X 6400 X 150 X 3}$

= ₹ 2400

Therefore Optimal Cost = ₹ (76800 + 2400)

= ₹ 79200

Example 5

Radhika motors purchases 18000 motor spare parts as its annual requirement, ordering one month usage at a time. Each spare part costs ₹ 20. The ordering cost per order is ₹ 15 and the carrying charges are 30% of the average inventory per year.

You have been asked to suggest a most economical purchasing policy for the company. What advice would you offer, and how much would it save the company.

Solution:- Annual Consumption (D) = 18000 units

Cost per unit = ₹ 20

Ordering cost per order (Co)= ₹ 15

Carrying Cost per unit (Cc) = (20X30%) = ₹ 6

Present Policy

Units per order = D/Number of orders

= 18000/12

= 1500 units

Ordering Cost = Number of orders X Ordering Cost per order

= 12 X 15 = ₹ 180

Carrying cost per unit = $\frac{\text{Units per order}}{2}$ X Carrying cost per unit

= (1500/2) X 6

= ₹ 4500

Total variable Cost = Ordering cost + Carrying cost

= 180 + 4500

= ₹ 4680

IF EOQ is followed

EOQ = $\sqrt{2DCo/Cc}$

= $\sqrt{\frac{2 X 18000 X 15}{6}}$

EOQ = 300 units

Ordering Cost = Number of orders X ordering cost per order

Number of orders = $\frac{\text{Annual Demand}}{\text{EOQ}}$

= 18000/300 = 60 orders

Ordering cost = 60 X 15 = ₹ 900

Carrying Cost = (EOQ/2) X carrying cost per unit

= (300/2) X 6 = ₹ 900

Total variable Cost = Ordering cost + Carrying cost

= 900 + 900

= ₹ 1800

Alternatively

Total Variable Cost = $\sqrt{2\ D\ Co\ Cc}$

= $\sqrt{2\ X\ 18000\ X\ 15\ X\ 6}$

= ₹ 1800

Thus, the company is advised to purchase 300 units at a time instead of ordering 1500 units every month.

It will result in saving of ₹ (4680-1800) = ₹ 2880

B) **Tabular Method** :- Economic order quantity can also be calculated with the help of constructing a table. This method shall be clear from the following example:

Example 6

Annual Usage = 800 units

Cost of Material per unit = ₹ 20

Cost of placing and receiving one order = ₹ 50

Annual Carrying cost of inventory = 10% of inventory value

Determine EOQ using tabular method.

Solution :-

Annual Usage	No. of orders per year	Units per order or lot size	Ordering cost @ ₹ 50 per order	Average Inventory	Carrying cost @ ₹ 2 per unit	Total Variable Cost
A	B	C=A/B	D=B X 50	E= C/2	F= E X 2	G= D + F
800 units	1	800	50	400	800	850

	2	400	100	200	400	500
	3	267	150	133.5	267	417
	4	200	200	100	200	400
	5	160	250	80	160	410
	6	133	300	66.5	133	433
	7	114	350	57	114	464
	8	100	400	50	100	500
	9	89	450	44.5	89	539
	10	80	500	40	80	580

Table shows that the cost of placing and receiving order and carrying cost are equal when order is 200 units. Moreover the total variable cost of ₹ 400 is the lowest when the order quantity is 200 units or the number of orders in a year are 4.

Example 7

Novelty Ltd carries a wide assortment of items for its customers. One item Gaylook, is very popular. Desirous of keeping its inventory under control, a decision is taken to order only the optimal economic quantity, for this item, each time. You have the following information. Make your recommendations:

Annual Demand = 160000 units

Price per unit = ₹ 20

Carrying cost = ₹ 1 per unit or 5% per rupee of inventory value

Cost per order = ₹ 50

Determine the optimal economic order quantity by developing the following table.

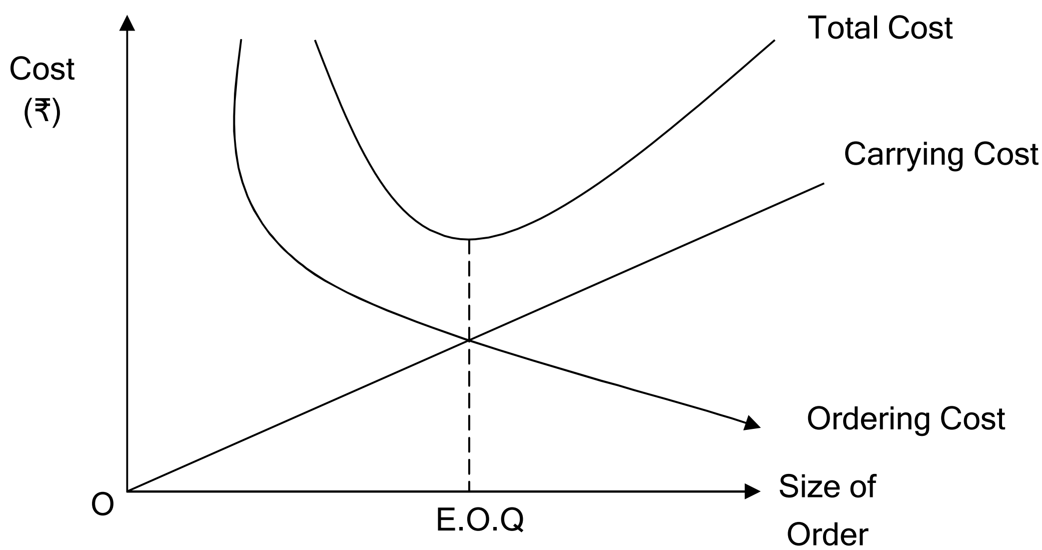
Number of orders	1	10	20	40	80	100
Average inventory, carrying cost, ordering cost, Total cost						

Solution :-

Annual Usage	No. of orders per year	Units per order or lot size	Ordering cost @ ₹ 50 per order	Average Inventory	Carrying cost @ ₹ 1 per unit	Total Variable Cost
A	B	C=A/B	D=B X 50	E= C/2	F= E X 1	G= D + F
160000 units	1	160000	50	80000	80000	80050
	10	16000	500	8000	8000	8500
	20	8000	1000	4000	4000	5000
	40	4000	2000	2000	2000	4000
	80	2000	4000	1000	1000	5000
	100	1600	5000	800	800	5800

Since the total cost per year is minimum when number of orders in a year are 40, therefore the EOQ is 4000 units.

C) **Graphic Method** :- Since Economic order quantity is that quantity of material where carrying costs and ordering costs are equal and the total cost is the lowest. Thus economic order quantity can be determined graphically as follows:



The above figure makes it clear when carrying costs and ordering costs are same, their graph intersects each other at point E and it is the point where economic order quantity occurs. Moreover at this point total cost of inventory is the minimum.

CHECK YOUR PROGRESS A

(i) E.O.Q is that order quantity where total cost is the maximum (True/False).

(ii) If unit cost = ₹ 20, Ordering cost per order = ₹ 5 and the carrying cost is 10% then carrying cost perunit is______________________.

(iii) Total variable cost on purchase of material is.

- (a) Ordering cost + carrying cost
- (b) Acquisition cost + ordering cost
- (c) Acquisition cost + carrying cost
- (d) Acquisition cost + ordering cost + carrying cost

(iv) Which of the following is not an assumption of E.O.Q Model?

- (a) Demand for the item is known and constant
- (b) If orders are placed at right time, there will not be any stock out position
- (c) The receipt of the inventory is instantaneous
- (d) The lead time is neither known nor constant

7.3 E.O.Q AND QUANTITY DISCOUNT

Sometimes, the suppliers offer discount for bulk purchases in order to achieve higher sales volume. Such discounts are called quantity discounts.

The discount in price affects inventory cost in three ways:-

(i) As the price per unit is reduced and hence total item cost also decreases.

(ii) When goods are purchased in large quantity, the number of orders is reduced resulting in reduction in ordering cost.

(iii) Since the quantity per order increases, the average inventory held goes up and hence holding cost/ carrying cost increases.

To make a decision in such a situation, the total cost of inventory without discount is compared with the total cost of inventory with quantity discount. In case the total inventory cost with quantity discount offer is less than total cost of inventory without discount, the offer is accepted. Otherwise offer is rejected.

Following steps are used in deciding E.O.Q with quantity discount.

Step 1 :- Calculate E.O.Q

Step 2 :- Calculate total cost including product cost for E.O.Q

Step 3 :- Calculate total cost including product cost for quantity at which discount is available

Step 4 :- Compare and decide

Example 8

Annual Demand for a particular item of inventory is 20,000 units. Inventory carrying cost per unit per year is 10% and the ordering cost is ₹ 20 per order. The price quoted by the supplier is ₹ 8 per unit.

Find out Economic order quantity and the number of orders in a year.

Would you accept 2% price discount on a minimum supply quantity of 1500 units? Compare the total cost in both the cases.

Solution:-

	Annual Demand (D)	=	20000 units
	Cost per unit	=	₹ 8
	Ordering cost per order (Co)	=	₹ 20
	Carrying Cost per unit (Cc)	=	(8 X 10%) = 0.8
	EOQ	=	$\sqrt{2DCo/Cc}$
		=	$\sqrt{\frac{2 X 20000 X 20}{0.8}}$
	EOQ	=	1000 units
	Number of orders in a year	=	Annual Demand / E.O.Q
		=	20000/1000

$= \quad 20$ orders

Total cost = Cost of items + Ordering cost + Carrying cost

$= (20000 \times 8) + (20 \times 20) + (1000/2) \times 0.8$

$= 160000 + 400 + 400$

= ₹ 160800

Average cost per unit $= 160800/20000$

= ₹ 8.04

In case of 2% price discount at minimum supply of 1500 units

Cost price per unit = 8 less 2% = ₹ 7.84

Carrying cost per unit (Cc) = $7.84 \times 10\% = 0.784$

Total cost = Cost of items + Ordering cost + Carrying cost

$=(20000 \times 7.84) + (20000/1500) \times 20 + (1500/2) \times .784$

$= 156800 + 267 + 588$

= ₹ 157655

Average cost per unit $= 157655/20000$

= ₹ 7.88

Since total annual cost and the per unit cost both are reduced when order is placed for 1500 units, discount offer should be availed of.

7.4 PURCHASE INVENTORY MODEL WITH ONE PRICE BREAK (SINGLE DISCOUNT)

It is an another way to decide whether to accept quantity discount offer or not. If a single discount is available we say there is one price break. Such a type of situation may be represented in the following manner:

Range of quantity	Price per unit
$1 \leq Q_1 < b$	P_1
$b \leq Q_2$	P_2

Here b = quantity beyond which discount is available and $P_2 < P_1$

Steps

1) Calculate Q_2 based on price P_2 by using the basic formula of E.O.Q i.e. $\sqrt{2DCo/Cc}$
2) If $Q_2 \geq b$ then order for Q_2 as it is the optimum order quantity and calculate optimal cost TC associated with Q_2 by adding material cost, ordering cost and carrying cost. If $Q_2 < b$ then go to Step 3
3) Calculate Q_1 for P_1. Calculate TC for Q_1 and TC for b
4) If Total cost (Q_1) > Total cost (b) then order Quantity is b.

Example 9

Find the optimum order quantity for a product for which the price breaks are as follows:

Quantity	Unit Cost
$1 \leq Q_1 < 800$	20
$800 \leq Q_2$	18

The annual demand for the product is 1600 units. Ordering cost is ₹ 5 and cost of storage is 10% per annum.

Solution:-

1st Step :- Calculate Q_2 based on price P_2

$P_2 = 18$

Carrying Cost (Cc) = (18 X 10%) = 1.8

$$Q_2 = \sqrt{2DCo/Cc}$$

$$= \sqrt{\frac{2 X 1600 X 5}{1.8}}$$

$$Q_2 = 94$$

Since Q_2 is less than b i.e. 800 units, thus it is not feasible.

2nd Step :- Now consider P_1 and hence optimum order quantity Q_1 is calculated as

$Q_1 \quad = \quad \sqrt{\frac{2 X 1600 X 5}{10\% \; of \; 20}}$

$= \quad \sqrt{\frac{2 X 1600 X 5}{2}}$

= 89.4 units or 89 units

3^{rd} Step :-

Total cost for Q_1 = Material cost + ordering cost + carrying cost

= (1600 X 20) + (1600/89) X 5 + (89/2) X 2

= 32000 + 89.89 + 89

= ₹ 32178.89 = ₹ 32179 (approx)

Total cost for b = Material cost + ordering cost + carrying cost

= (1600 X 18) + (1600/800) X 5 + (89/2) X 1.8

= 28800 + 10 + 720

= ₹ 29530

Since TC (Q_1) > TC (b), the optimum purchase quantity is 800 units.

Example 10

Find EOQ

D= 3600 units

Co = ₹ 50

Cc = 20% of unit cost

Quantity	Unit Cost
$1 \leq Q_1 < 100$	20
$100 \leq Q_2$	18

Solution:-

1^{st} Step :- Calculate Q_2 based on price P_2

$Q_2 \quad = \quad \sqrt{2DCo/Cc}$

$$= \sqrt{\frac{2 X 3600 X 50}{18 X 20\%}}$$

$$= \sqrt{\frac{2 X 3600 X 50}{3.6}}$$

$$Q_2 = 316 \text{ units}$$

Since $Q_2 > b$, it is feasible and hence the purchase quantity is 316 units.

7.5 PURCHASE INVENTORY MODEL WITH MULTIPLE PRICE BREAKS (MULTIPLE DISCOUNT)

Sometimes more than one price break may be offered by the supplier. This is simply an extension of the mechanism in which single price break is offered. The situation can be represented as below:-

Quantity	Price per unit (₹)
$1 \leq Q_1 < b_1$	P_1
$b_1 \leq Q_2 < b_2$	P_2
$b_2 \leq Q_3 < b_3$	P_3
---------	---
$b_{n-1} \leq Q_n$	P_n

<u>Steps</u> :-

1) Determine EOQ with the lowest price available. If it is feasible then it is the optimum quantity.
2) If EOQ is less than the quantity at which relevant discount offer is given, then it is not feasible. Hence calculate EOQ at next lowest price. If this EOQ falls in the range of the lot size of which the price is used, then we are having optimum solution. Again, if it is not feasible, then calculate EOQ at next lowest price available.
3) Continue the steps until a feasible EOQ is obtained, then determine total cost corresponding to this quantity level and the subsequent lot size categories.
4) Compare the total cost so determined at various cut off points and select the one with minimum cost.

This procedure involves only a finite number of steps, at the most n, where n is the number of price ranges offered by the supplier.

Example 11

Determine the best order size from the data given below:

Quantity	Unit Cost
1-999	22
1000-1499	20
1500-1999	19
2000 and above	18.50

Annual consumption is 8000 units, carrying cost 10% of average inventory and the ordering cost is ₹ 180 per order.

Solution:-

1st Step :- Calculate optimum quantity Q_4 at price ₹ 18.50

$$Q_4 = \sqrt{2DCo/Cc}$$

$$= \sqrt{\frac{2 X 8000 X 180}{18.50 X 10\%}}$$

$$Q_4 = 1248 \text{ units}$$

Since $Q_4 < 2000$, so availing this price in not feasible.

2nd Step :- Calculate optimum quantity Q_3 at price ₹ 19

$$Q_3 = \sqrt{2DCo/Cc}$$

$$= \sqrt{\frac{2 X 8000 X 180}{19 X 10\%}}$$

$$= 1231 \text{ units}$$

However to avail price after of ₹ 19, the quantity purchased should be at least 1500 units.

3rd Step :- Calculate optimum Quantity Q_2 at price ₹ 20

$$Q_2 = \sqrt{2DCo/Cc}$$

$$= \sqrt{\frac{2 X 8000 X 180}{20 X 10\%}}$$

$$= \sqrt{\frac{2 X 8000 X 180}{2}}$$

= 1200 units

Since Quantity Q_2 is within the range of 1000-1499 at which price of ₹ 20 per unit is offered, so it is feasible to avail this price offer.

4^{th} Step :-

Hence Total cost (Q_2) = Material cost + ordering cost + carrying cost

= (8000 X 20) + (8000/1200) X 180 + (1200/2) X 2

= 160000 + 1200 + 1200

= ₹ 162400

Total cost (b_2 i.e. 1500 units) = Material cost + ordering cost + carrying cost

= (8000 X 19) + (8000/1500) X 180 + (1500/2) X 1.9

= 152000 + 960 + 1425

= ₹ 154385

Total cost (b_3 i.e. 2000 units) = Material cost + ordering cost + carrying cost

= (8000 X 18.50) + (8000/2000) X 180 + (2000/2) X 1.85

= 148000 + 720 + 1850

= ₹ 150570

Since TC (b_3) < TC (b_2) < TC (Q_2), thus optimum quantity to purchase is b_3 i.e. 2000 units.

Example 12

Determine the optimal order quantity for a product for which price break are as follows:

Quantity	Price per unit (₹)
$1 \le Q_1 < 600$	11
$600 \le Q_2 < 750$	9.75
$750 \le Q_3$	8.75

The annual demand for the product is 200 units, holding cost is 2% of unit cost and ordering cost is ₹ 350.

Solution:-

Calculate optimum quantity Q_3 at price ₹ 8.75

$$Q_3 = \sqrt{2DCo/Cc}$$

$$= \sqrt{\frac{2 \times 200 \times 350}{8.75 \times 2\%}}$$

$$= \sqrt{\frac{140000}{0.175}}$$

$$Q3 = 894 \text{ units}$$

Since Q_3 is > 750 thus it is feasible to avail this price offer. Hence optimal order quantity is 894 units.

CHECK YOUR PROGRESS B

(i) Choose the correct option.

Decision Rule in case of discount offer

(a) If Total cost (Discount) ≤ Total Cost (EOQ) --- Accept discount offer

(b) If Total cost (Discount) > Total Cost (EOQ) --- Accept discount offer

(ii) Quantity discount leads to

(a) No change in cost

(b) Decrease in total cost

(c) Increase in ordering cost

(d) Decrease in carrying cost

(iii) Unit price offered for quantity more than 100 units is ₹ 18, thus if quantity determined is 200 units, then it is_______________________to avail price offer of ₹ 18.

(iv) With the decrease in unit price, the carrying cost per unit will ___________.

7.6 SUMMARY

EOQ is a very important tool in the hands of management to make a decision for how much and when to order in such a way that total cost is minimum and there is a balance between ordering cost and carrying cost. An extension to simple EOQ model helps to make effective decision for making purchases under the conditions when different prices are offered for varied range of quantities.

7.7 GLOSSARY

Ordering Cost :-	It is the cost of placing an order and securing the supplies e.g. rent of the place used by purchase department, salaries of purchase department, stationery, transportation, inspection cost etc.
Carrying Cost :-	The cost of keeping the material in store house e.g. capital cost, cost of storage, cost of deterioration, store insurance etc.
Lead Time :-	The time gap between placing the order and receiving the supplies.
Quantity Discount :-	A discount offered for bulk purchases.

7.8 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS A

(i) False
(ii) ₹ 2
(iii) (a)ordering cost + carrying cost
(iv) (d)The lead time is neither known nor constant

CHECK YOUR PROGRESS B

(i) (a)If Total cost (Discount) ≤ Total Cost (EOQ) --- Accept discount offer
(ii) (b)decrease in total cost
(iii) feasible
(iv) decrease

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7.11 TERMINAL AND MODEL QUESTIONS

1) What is Economic order quantity? How is it calculated? Discuss various assumptions of E.O.Q

2) Discuss the steps involved in purchase inventory model with one price break and multiple price breaks.

3) The annual requirement for a particular raw material is 2000 units costing ₹ 1 per unit. The ordering cost is ₹ 10 per order and the carrying cost is 16% per annum of the average inventory value.

Find (a) E.O.Q

(b) Number of orders per year

(c) Time between orders

(d) Total inventory cost per annum

4) The purchase department of your organization has received an offer of quantity discounts on its orders of materials as next:

Price per Tonne (₹)	Tonnes
1200	Less than 500
1180	500-1000
1160	1000-2000
1140	2000-3000
1120	3000 and above

The annual requirement for the material is 5000 tonnes. The ordering cost per order is ₹ 1200 and the stock holding cost is estimated @ 20% of material cost per annum.

You are required to advise the purchase department the most economical purchase level

7.12 ACTIVITY

Annual Demand in a manufacturing unit is 600 units, ordering cost is ₹ 400, holding cost is 40%, and cost per unit is ₹ 15. A supplier offers a discount of 10% on the order quantity of 500 units. You are requested to give advise to the manufacturing unit whether to avail discount offer or not. Also give justification for your advise.

Chapter 8: PROJECT MANAGEMENT

8.0 OBJECTIVES

This chapter should help students to understand:

- Important of managing a project
- Network techniques of performing project management functions
- Rules of drawing a network diagram
- Finding of critical and non-critical activities by using Critical Path Method (CPM)
- Finding probability of completing a project by using Program Evaluation and Review Technique (PERT)
- Finding minimum time of project completion with impact on project cost

8.1 INTRODUCTION

Building of a house requires carrying out numerous activities sequentially as well as simultaneously. It is necessary to identify type of activities and resources required to carry out such activities. Some of such activities might be creating foundation, construction, wood work, electric work, interior designing etc. Each activity can be further broken down into number of activities. Thus, a project in this example construction of house involves combination of number of activities arranged in such a sequence so as to achieve completion of project in significant period of time. Definition of project involves following important aspects:

- *Identification of activities:* which activities need to be performed for completion of project?
- *Sequence of activities:* which activities are to be performed in what sequence and can some of them be performed simultaneously or can some of them be combined?
- *Resources required:* each activity requires certain resources in terms of man, machine, material, time and effort. How many individuals are required to perform a particular activity? Can they be used for performance of other activities? How much time does each activity take? What is the cost required to perform each activity?
- *Critical activities:* which are the most important activities and what is the time required to complete those activities? Can these activities be delayed and if yes what would be the effect on project completion time? Which are non-critical activities or activities which can be delayed without delaying the project?

Thus, a project involves number of aspects and its successful completion requires a scientific and systematic approach. PERT and CPM are two of the such approaches which answers above mentioned questions. These two methods are network oriented techniques used primarily to determine project completion time. The basic difference between two techniques is that PERT is probabilistic technique where time required for carrying out an activity is not known with certainty. CPM is a deterministic time where completion time of each activity is known with certainty. CPM is also used to find out critical and non-critical activities. The identification of such activities helps a manager to evaluate which activities can be delayed and which cannot with an effect on project completion time. This chapter discusses these two network methods in detail with illustrations. Also, chapter discusses time-cost model of project which provides the effect of change in time of activity on total cost of completion of project. This chapter is limited to study of estimation of project completion time and impact of time as a resource on project cost.

8.2 PROJECT MANAGEMENT: IMPORTANCE

Project management primarily involves following three managerial functions:

- ***Planning:*** involves identification and breaking down of activities into viable and feasible tasks that must be performed for completion of project. It also involves determination and allocation of resources to each activity. It requires estimation of number of man hours, amount of material required, cost that would be incurred etc. for each activity.
- ***Scheduling:*** involves arranging activities in a logical sequence that would help in converting allocated resources to desired outputs in most productive manner. It is important to note that identified activities can be arranged in different ways to obtain certain outputs. But it is important to find out the most productive way of arranging activities. Such approach would also require identification of critical activities and estimation of available idle time for which activities can be delayed.

- ***Control:*** function involves review of project progress. This function determines whether each activity is performed in estimated duration, uses allocated resources or is deviating from planned schedule. This function allows controlling any difference between planned and actual schedule. The reasons for such a difference are analyzed and remedial measures adopted.

PERT and CPM methods of project management helps management to achieve these functions effectively as these methods identifies and estimates time for performance of activities, arranges them in most efficient sequence and provides data regarding any deviation from planned schedule. Following is an illustration about above discussed functions of project management.

In establishing an automatic car wash station project manager identifies three activities required to wash a car. Thus in first step it was determined that car wash involves scrubbing, rinsing and drying as three activities. These activities cannot be broken down further as that would make activities unviable in the sense it would take more resources and give less output. Also, time required to perform each activity is estimated. In this case time require for scrubbing, rinsing and drying was estimated at 2, 4 and 2 minutes respectively per car. Secondly and most importantly project manager has to decide sequence of performing identified activities. The sequence is decided by logic of technological process i.e. rinsing cannot be performed before scrubbing and neither drying can be carried out before these operations. Also, can identified activities be performed simultaneously i.e. in this case can a car go through scrubbing or rinsing, rinsing or drying and scrubbing or drying at same time. Again technological sequence limits such a possibility. Thus project manager decides sequence for washing a car as shown in table 8.2.1.

Table 8.2.1		
Scrubbing $\longrightarrow$	Rinsing $\longrightarrow$	Drying
2 minutes	4 minutes	2 minutes

Lastly project management network techniques such as PERT and CPM would help project manager to determine whether each car is being washed in estimated time of 8 minutes. Thus, these network techniques are important in estimating project schedule and completion time. The first step in application of these techniques is creation of network diagram involving logical sequence of activities. Following section discusses steps involved in construction of network diagram.

8.3 NETWORK CONSTRUCTION: SOME DEFINITIONS

Network is a web of identified activities showing their arrangement or sequence in a diagrammatic fashion. Construction of network diagram requires understanding of following terms:

Activity: An activity is a task that requires certain resources such as time, money and labor for its completion. In a network diagram it is represented by an arrow. An activity occurs over a period of time.

Event: An event indicates start or finish of an activity. It is represented by a circle called as node. An event does not consume any resource and it occurs at a point in time.

If in a network diagram an activity is represented by arrow and an event by node then such a diagram is called as activity on arrow (A-o-A) network diagram. Whereas if in a network diagram an activity is represented by a node and an event by an arrow then such a diagram is called as activity on node (A-o-N) network diagram.

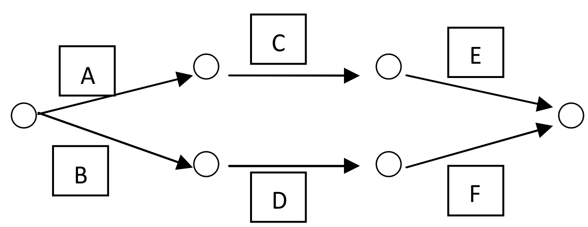
Predecessor: An activity which has to be completed before the start of another activity is called as predecessor to start of new activity.

Successor: An activity which starts after or succeeds the finish of another activity is called successor or succeeding activity.

Concurrent activity: An activity which can be performed simultaneously with some other activity is called as concurrent or parallel activities.

These activities are illustrated in figure 8.3.1. This is an activity on arrow network diagram. A is preceding activity to C which is preceding to E. Similarly, B is preceding to D which is preceding to F. Activities can also be explained in successor terms. E activity will start after finish of C i.e. it succeeds C which succeeds A. Similarly, F succeeds D which succeeds B. Events are depicted by circles called as nodes. They indicate the start and finish of an activity. It is important to note that A and B start from same node at same time making them concurrent or parallel activities.

Fig. 8.3.1



8.3.1 NETWORK CONSTRUCTION: RULES

A project broken down into six activities and time taken to complete each activity as shown in table 3.3.1.1 is considered for illustrating some rules used in construction of network diagrams

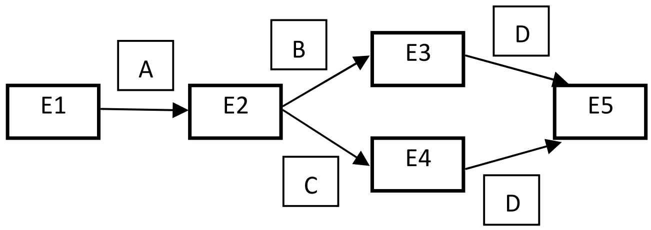
Table 8.3.1.1

Activity	A	B	C	D	E	F
Predecessor	----	A	A	B, C	B	D, E
Duration (days)	6	9	8	4	6	6

Rule 1:

Each activity is represented by only one arrow. In activity on arrow network diagram an arrow connecting two events can represent at the most one activity.

Fig. 8.3.1.1

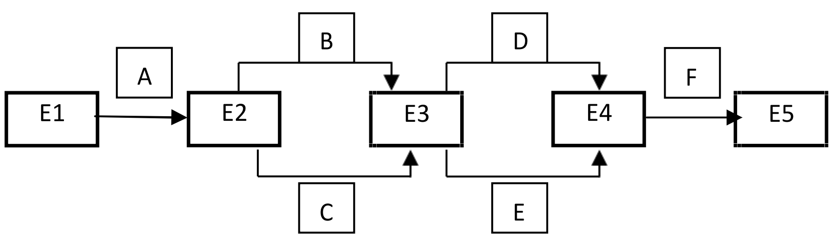


E1, E2, E3, E4 and E5 represent events and A, B, C and D represents activities. According to precedence table D is preceded by B and C. The network diagram shown in Fig. 8.3.1.1 is incorrect as it violates rule 1. According to rule one activity can be represented only by one arrow but D is represented by two arrows to fulfill precedence criteria. So, network diagram of Fig. 8.3.1.1 is incorrect.

Rule 2:

In order to fulfill precedence criteria the preceding, succeeding and any parallel activity to a particular activity must be ascertained. Fig. 8.3.1.2 shows start of B and C activity is happening simultaneously at event E2. E3 indicates finish of these two activities and also start of D and E. From the network diagram it can also be interpreted that B and C are preceding activities to D and E. But according to project tasks as shown in table 8.3.1.1 E has only one predecessor i.e. B. Thus, it is important to clearly understand precedence rule before constructing a network diagram.

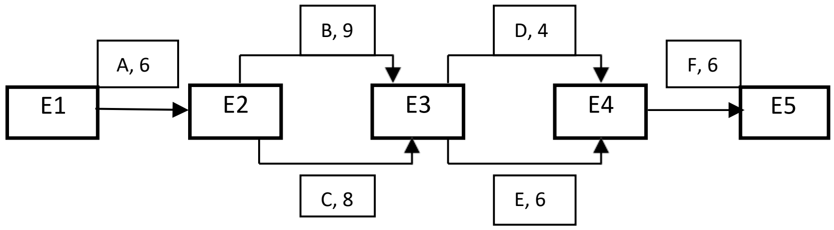
Fig. 8.3.1.2



Rule 3:

The network diagram must follow the logical sequence of activities i.e. an activity cannot start until its preceding activities on which it depends have been completed. For instance in car wash example drying cannot start until rinsing of a car has been completed which cannot start until scrubbing of that car has been completed. The logical sequence depends on precedence rule and time taken by each activity. E3 event in Fig. 8.3.1.3 shows finish of B and C and start of succeeding activities. But these activities cannot start until B and C both gets completed. Also B and C should not be shown as parallel activities. Though both starts from same event i.e. E2 but as both takes different times for completion so they are not parallel activities but can happen simultaneously.

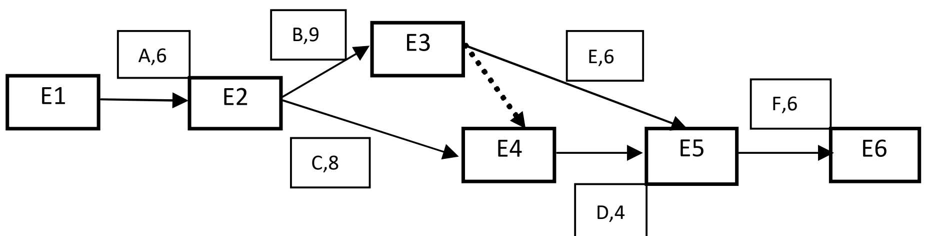
Fig. 8.3.1.3



Dummy Activities:

Dummy activities are used in activity on arrow network diagrams to resolve some of the issues as discussed above. The purpose of dummy activity is to follow precedence relationship without violating any of the rules of network construction.

Fig. 8.3.1.4



A dummy activity does not consume any resources in terms of time and effort and is solely used as connector of two or more activities when they have same start and finish nodes. The example shown in table 8.3.1.1 indicates that both B and C have same starting event and finish event. Both are succeeded by same activity i.e. D making it an appropriate case of applying dummy activity as shown in Fig. 8.3.1.4. Dummy activity is shown as dotted line in the figure. It can be observed that by applying dummy activity all rules of network construction are fulfilled.

Exercise 1

1. Activities are represented as arrow on__________diagram and as nodes on__________diagram.
2. A network diagram is a combination of_________arranged in a_________sequence.
3. Which of the following are characteristics of a project?
   (a) it is non-repetitive
   (b) it has a beginning and an end
   (c) it can be broken down into finite activities
   (d) all of the above
4. Dummy activities:
   (a) are used as connector
   (b) does not consume any resources
   (c) are drawn to connect two activities when they have same start and finish nodes

(d) all of the above

5. A network diagram:
   (a) depicts relationship between activities
   (b) activities are arranged in a sequence of precedence or succession
   (c) arrangement of activities is governed by technological and process sequence
   (d) all of the above

8.4 <u>CRITICAL PATH METHOD (CPM)</u>

CPM is a deterministic approach of network technique where duration of each activity involved in completion of project is known with certainty. Certain activity times would be known in cases of projects which are repetitive or are not new for an organization. For example an IT firm specializing in creating software for different banks would apply CPM for a new banking project. As company has already created software for other banks so creating software for new bank would involve similar tasks for which completion time is known with high certainty.

The purpose of CPM technique is to:

- Find scheduled project completion time.
- Scheduled start and finish times for each activity.
- Identification of most important or critical activities which should be completed within scheduled time without delaying the project.
- Identification of non-critical activities which can be delayed by specific time without delaying the completion of project from its scheduled time.

To achieve objectives of CPM certain terms have to be understood:

Earliest Start Time (EST) of an activity indicates the time at which an activity can start as soon as possible.

Earliest Finish Time (EFT) of an activity indicates the earliest possible time at which an activity can be completed. It is the sum of EST and duration of that activity.

Calculation of EST is governed by *forward path* rule. This rule states that for calculating EST we should move forward from first event to the last event in a network diagram. As an activity cannot start until all preceding activities have been completed so EST of an activity is equal to the largest EFT of immediate preceding activity.

Latest Finish Time (LFT) of an activity indicates finish of an activity as late as possible without having an effect on scheduled finish of entire project.

Latest Start Time (LST) of an activity indicates start of an activity as late as possible without having an effect on scheduled finish of entire project. LST is the difference of LFT and duration of particular activity.

Calculation of LFT is governed by *backward path* rule. This rule states that for calculating LFT we should move backwards from last event to the first event in a network diagram. As purpose is to finish an activity as early as possible so LFT for an activity is the smallest of latest start time of immediate succeeding activities.

Slack is determined after finding out all earliest and latest times by moving forward and backward in a network diagram. Slack indicates idle time of an activity with which it can be delayed without increasing the project completion time. It can be computed as: LFT – EFT or LST – EST.

Following notation would be used to represent discussed times.

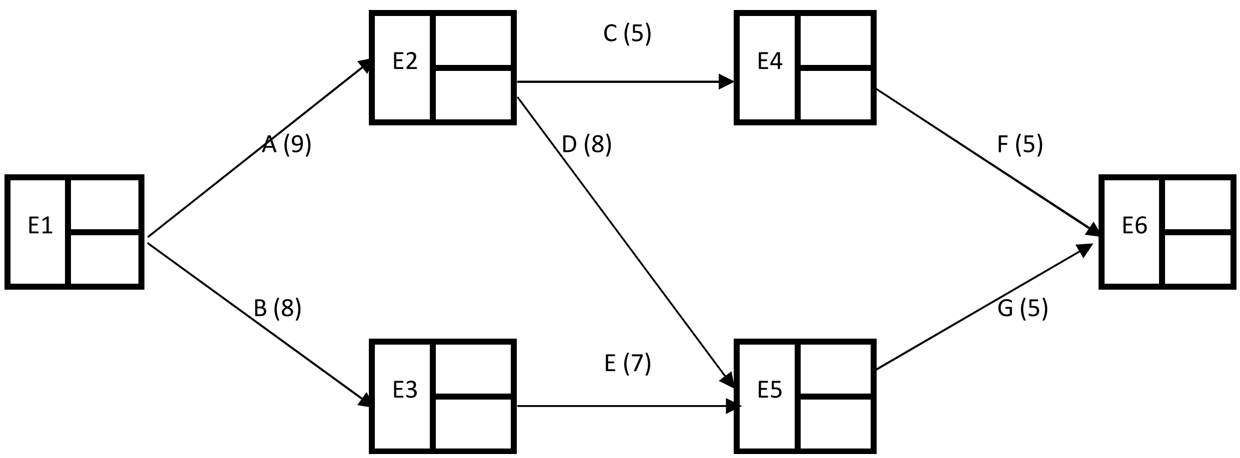
Event	**EST of succeeding activity**
	LFT of preceding activity

Example: A project manager has identified following activities, their logical sequence and their duration times in days for completion of a project. How CPM technique can help the manager to plan, schedule and complete the project?

Activity	A	B	C	D	E	F	G
Activity time (days)	9	8	5	8	7	5	5
Predecessor	----	----	A	A	B	C	D, E

Step 1: Construct network diagram:

Network diagram is constructed according to logical sequence as shown in following figure.



Step 2: Calculation of scheduled times:

Activity	**Activity time (t, days)**	**EST**	**EFT (EST + t)**	**LFT**	**LST (LFT – t)**	**Slack (LST – EST)**	**Critical Path**
A	9	0	9	9	0	0	Yes
B	8	0	8	10	2	2	---
C	5	9	14	17	12	3	---
D	8	9	17	17	9	0	Yes
E	7	8	15	17	10	2	---

F	5	14	19	22	17	3	---
G	5	17	22	22	17	0	Yes

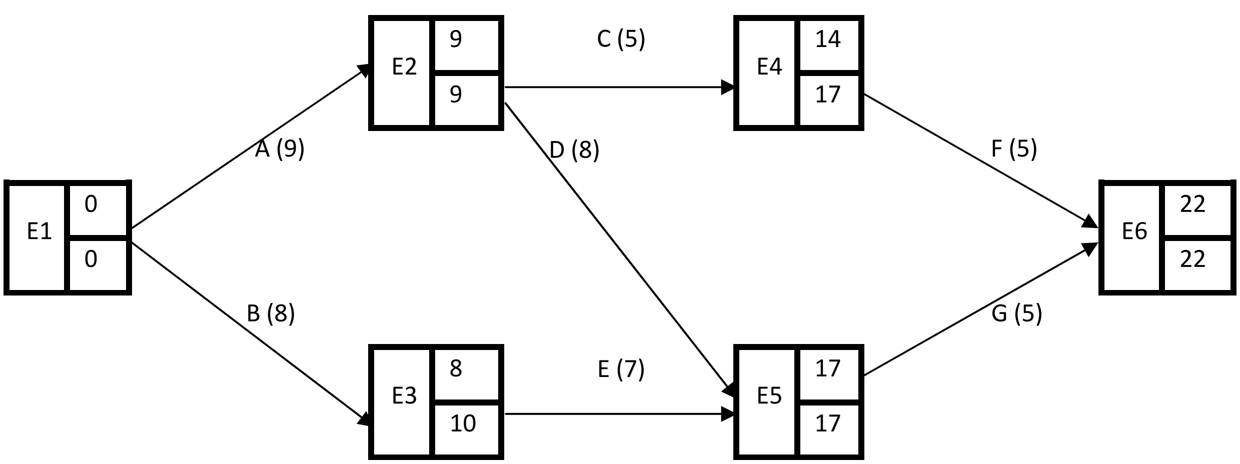
EST is calculated by moving forward in the network diagram. Activity A starts at E1 and it can start at the earliest i.e. at 0 time. Activity C starts after finish of A which happens after 9 days so it's EST is 9 days. Now what should be EST of G. Activity G will start only after finish of activities A, B, D and E indicated by event E5. Now E5 can be reached by moving along paths of A---D and B---E. Total completion time of path A---D would be 9 + 8 = 17 days whereas that of path B---E would be 8 + 7 = 15 days. As G would not start till all preceding activities are finished so its start time would be 17 days.

LFT is calculated by moving backwards from last event in the network diagram. E6 indicates finish of all the paths. Now E6 can be reached by moving along A---C---F, A---D---G and B---E---G. Total time for each of the path would be 19 days, 22 days and 20 days respectively. As project cannot be finished till all activities are completed so LFT of E6 would be 22 days. Moving backwards C finishes at E4. Its LFT would be 22-5 = 17 days and so on. Interestingly, finish of E2 can be achieved by moving backwards from F to C or from G to D. For F to C LFT would be 22-5-5 = 12 days and for G to D LFT would be 22-5-8 = 9 days. As an activity should be finished as early as possible so by moving backwards minimum LFT should be considered. So finish of E2 or LFT of activity A would be 9 days. All other times are calculated and shown in table.

<u>Step 3:</u> Calculation of slack:

Zero slack of an activity would indicate that such an activity cannot be delayed or it has to be completed within scheduled time. Such activities are most important activities in a project and constitute critical path. In this case A---D---G constitute critical path which would take 22 days. This would also be the project completion time i.e. project comprising of above mentioned activities would take minimum of 22 days. Activities with some slack are called as non-critical activities. For instance, activity B with a slack of two days implies that even if start of this activity is delayed by two days then also project would be completed in 22 days.

Final network diagram is shown as:



8.5 PROGRAM EVALUATION AND REVIEW TECHNIQUE (PERT)

PERT is a probabilistic network technique used for planning, scheduling and estimating completion time of project. This technique is useful where activity times are not known with certainty. Some projects are new for a company for which it becomes difficult to establish certain times. In such cases where activity times are uncertain these are described by range of possible times rather than a single deterministic time. Following three time estimates are considered to find possible range of times for uncertain activities.

- ***Optimistic time (t_o):*** is the time taken by an activity when performed under ideal conditions. This is the minimum time taken as conditions are most favorable for its performance.
- ***Most likely time (t_m):*** is the time taken by an activity when performed under normal conditions. This is the most possible time under which an activity would be performed as conditions are most realistic.
- ***Pessimistic time (t_p):*** is the time taken by an activity when performed under worst conditions. This is the maximum time taken by an activity as it is performed when significant delays are most probable.

So, $t_o < t_m < t_p$

The average time of an uncertain activity in a new project is calculated by finding average of three probable activity times discussed above. This average time is called as expected time (t_e). Maximum weight age is given to tm as this is the most probable time an activity would take to perform. Thus,

$$t_e = (t_o + t_m + t_p) / 6$$

Because of uncertain times the variance in them can be calculated as:

$$\text{Variance} = [(t_p - t_o) / 6]^2$$

In light of above discussion major purposes of PERT are:

- Estimation of expected time of each activity
- Estimation of scheduled completion time of project
- Finding out critical and non-critical activities
- Most importantly PERT is used to find probability that whether project would complete in given scheduled time.

Above mentioned objectives are achieved by following below mentioned steps:

Step 1: Draw network diagram of the project.

Step 2: Find expected time and variance of each activity

Step 3: Find critical path by calculating EST, EFT, LST and LFT of each activity.

Step 4: Compute expected time and variance of critical path.

Step 5: Find standard normal deviation by using formula

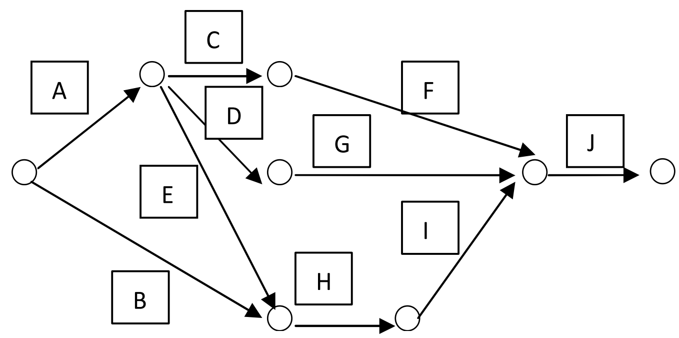
$$z = (\text{scheduled time} - \text{expected time of project}) / \text{standard deviation of critical path}$$

By using this z value from z tables probability of project completion within scheduled time can be evaluated.

Example: For the following project estimate probability of completion of project within 20 weeks.

Activity	A	B	C	D	E	F	G	H	I	J
Predecessor	---	---	A	A	A	C	D	B, E	H	F,G,I
Optimistic time (weeks, t_o)	4	1	2	3	2	1.5	1.5	2.5	1.5	1
Most likely time (weeks, t_m)	5	1.5	3	4	3	2	3	3.5	2	2
Pessimistic time (weeks, t_p)	12	5	4	11	4	2.5	4.5	7.5	2.5	3

<u>*Step 1:*</u> Following figure shows the network diagram



<u>*Step 2:*</u> Expected time and variance of each activity is shown ion following table

Activity	**Predecessor**	**Optimistic time (weeks, t_o)**	**Most likely time (weeks, t_m)**	**Pessimistic time (weeks, t_p)**	**Expected time (t_e)**	**Variance**
A	---	4	5	12	6	1.78
B	---	1	1.5	5	2	0.44
C	A	2	3	4	3	0.11
D	A	3	4	11	5	1.78
E	A	2	3	4	3	0.11
F	C	1.5	2	2.5	2	0.03
G	D	1.5	3	4.5	3	0.25
H	B, E	2.5	3.5	7.5	4	0.69
I	H	1.5	2	2.5	2	0.03
J	F,G,I	1	2	3	2	0.11

<u>*Step 3:*</u> Calculation of critical path:

Activity	**Expected time**	**EST**	**LST**	**EFT**	**LFT**	**Slack**

	(t_e)					
A	6	0	0	6	6	0
B	2	0	7	2	9	7
C	3	6	10	9	13	4
D	5	6	7	11	12	1
E	3	6	6	9	9	0
F	2	9	13	11	15	4
G	3	11	12	14	15	1
H	4	9	9	13	13	0
I	2	13	13	15	15	0
J	2	15	15	17	17	0

Thus critical path is: A (6) ----E (3) ----H (4) ----I (2) ---J (2) = 17 weeks

Step 4: Calculation of expected project time and variance:

Thus expected project duration is 17 weeks.

Variance of activities involved in critical path would be

Variance = 1.78 + 0.11 + 0.69 + 0.03 + 0.11 = 2.72

Standard deviation = √2.72 = 1.65

Step 5: Find standard normal deviation by using formula

z = (scheduled time – expected time of project) / standard deviation of critical path

z = (20 – 17) / 1.65

= 1.82

From z tables probability value corresponding to 1.82 standard deviation representing area between 20 and 17 weeks would be 0.4656. As project manager wants to find out probability of area less than 20 weeks i.e. probability of project getting completed in any time which is less than 20 would be

0.4656+0.5000 = 0.9656.

Thus, there is 96.56% chance that project will finish before 20week deadline.

Exercise 2:

1. PERT is a__________technique.
2. CPM is a__________technique
3. PERT uses which of the following three times to find expected time of each activity

   (a) optimistic time (b) pessimistic time (c) most likely time (d) all of the above
4. CPM is not used to find:

a) project completion time

b) probability of project completion within scheduled time

c) Critical and non-critical activities

5. Forward pass used in PERT/CPM helps in finding

a) EST (b) LST (c) EFT (d) LFT

8.6 CRASHING

The major application of CPM is to estimate the impact on total cost associated with project if project completion time is reduced. The knowledge of scheduled start and finish times of each activity, slack of each activity and identification of critical and non-critical activities obtained from application of CPM is used in estimating cost of project with change in project completion time. In network techniques of PERT and CPM the resource used with each activity is the duration of that activity. But performance of each activity also uses other resources such as labor, materials etc. which incur cost. Crashing method uses this information of cost associated with performance of each activity to achieve the objective of estimating project time. It is important to understand that with reduction in activity time more resources would be required for its completion thus, increasing its cost. That is why there would always be a trade-off between time and cost.

So, question that a project manager faces:

- Which activities to be crashed i.e. duration time of which activities should be reduced so that project time gets reduced?
- What is the increase in cost of project completion with reduction in time period of identified activities?
- For how much time period an activity can be crashed?

Crashing should not be done indefinitely. A stage will come when further crashing of activities will not reduce total project completion time but it would increase cost significantly. As illustration discussed below would show that with decrease in time of an activity it increases direct cost such as cost of machinery, labor etc. and reduces indirect cost such as cost of lighting, rent, security etc. At first total cost would reduce with reduction in project time because proportion of increase in direct cost is less than decrease in indirect cost. But after certain reductions in time period proportion of increase in direct cost would outweigh reduction in indirect cost thus increasing total cost. This is an indication that further crashing would not result in optimal results. Thus objective of crashing gets defeated. So, a project manager should stop crashing when all the paths in network diagram become critical or total cost starts increasing.

Following are certain terms that must be understood in application of crashing:

- *Normal time:* It is the expected time of completion of an activity.
- *Crash time:* It is the reduced time of completion of an activity.
- *Normal cost:* is the cost of performance of an activity when performed under normal conditions.
- *Crash cost:* is the cost of performance of an activity when performed under reduced time.

Following are the steps to be used in crashing of a project:

Step 1: Draw network diagram

Step 2: Find cost slope of each activity. Cost slope is the increase in *direct cost* of an activity when its duration is reduced by one day. Cost slope = change in cost / change in time

Step 3: Find critical path. As discussed critical path is found by calculating EST, EFT, LST and LFT. By definition critical path is the longest path in the network diagram. In example shown under critical path is estimated by finding the longest path in diagram.

Step 4: In the critical path find an activity with least cost slope because aim is to reduce project time with least increase in cost.

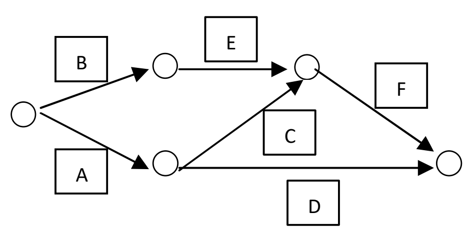
Step 5: Estimate the time for which selected activity can be crashed. For this a general rule that is followed is that activity should be crashed for so much time that it does not result in change of critical path.

Step 6: Keep on repeating the process till further reduction in time period leads to increase in cost or when all paths in network diagram have become critical.

Example: For the following project estimate total project cost under normal times and crash times. Indirect cost is Rs.100 per day.

Activity		**Normal**		**Crash**		**Cost slope**
	Predecessor	**Time (days)**	**Cost (Rs. 000)**	**Time (days)**	**Cost (Rs. 000)**	
A	---	8	100	6	200	50
B	---	4	150	2	350	100
C	A	2	50	1	90	40
D	A	10	100	5	400	60
E	B	5	100	1	200	25
F	C, E	3	80	1	100	10
		Total	***580***			

Step 1: Draw network diagram



Step 2: Find cost slope of each activity.

For instance cost slope of A = (200 – 100) / (8 – 6) = 50

Cost slope of B = (350 – 150) / (4 – 2) = 100 and so on…

Step 3: Find critical path. From start to finish all paths in network diagram are:

B---E---F = 12 days

A---C---F = 13 days

A---D = 18 days

Critical path by calculating the longest path would be A---D taking 18 days. Thus project completion time under normal conditions would be 18 days.

Step 4: In critical path i.e. A---D least cost slope is of activity A i.e. if duration of A is reduced by one day then increase in *direct cost* would be of Rs.50 per day.

Step 5: Estimate the time for which selected activity can be crashed.

(i) Activity A can be crashed by 2 days without making any other path critical.

So project time would be = 18 – 2 = 16 days.

Initial Total cost i.e. when project is completed in 18 days = Direct Cost + Indirect cost

= 580 + 18*100 = 580 + 1800 = ***Rs.2380***

When project is completed in 16 days

Increase in direct cost = 2*50 = Rs.100

Indirect cost = 16*100 = Rs.1600

So, Total cost = (580 + 100) + 1600

= ***Rs.2280***

(ii) Path A---D is critical with longest time duration of 16 days. As A is now being done in 6 days so it cannot be crashed further. D can be crashed by five days but if it is crashed by 5 days then path A---D would become non-critical which violates the rule i.e. crashing should be done in such a way that existing critical path should not become non-critical. So D should be crashed by 3 days. In such a way both A---D and A---C---F would become critical taking 13 days for project completion.

By crashing D by 3 days project completion time = 16 – 3 = 13 days.

Increase in direct cost = 3*60 = Rs.180

Indirect cost = 13*100 = Rs.1300

Total cost = (680 + 180) + 1300

= ***Rs.2160***

(iii) Now there are two critical paths A---D and A---C---F. In path A---D only D can be crashed further by 2 days with increase in cost of Rs.60 per day. In path A---C---F, activity F with least cost slope can be crashed by 2 days. But if both activities are crashed by 2 days then it again violates the rule of critical path i.e. it would make other path B---E---F as critical which is not acceptable. So crash both D and F by one day.

Project duration would be = 13-1 = 12 days

Increase in direct cost = 1*60 + 1*10 = Rs.70

Indirect cost = 12*100 = Rs.1200

Total cost = (780 + 70) + 1200

= ***Rs.2050***

(iv) Now all three paths are critical with 12 days as project duration. From path A---D activity D can be crashed further by 1 day with increase in cost of Rs.60 per day. From path B---E---F activity F can be crashed by 1 day with least cost slope of Rs.10 per day.

All three paths i.e. A---D, B---E---F and A---C---F are critical with every path taking 11 days for completion of project.

So project duration = 12 – 1 = 11 days

Increase in direct cost = 1*60 + 1*10 = Rs.70

Indirect cost = 11*100 = Rs.1100

Total cost = (850 + 70) + 1100

= ***Rs.2020***

Further crashing is not possible as one other paths i.e. A---D has reached its crash limit. Further crashing of any activity in any other path would violate rule of criticality.

8.7 RESOURCE LEVELING

Resource leveling is a network technique used to smoothen out the activities which are over allocated with certain resources. Activities are performed by using certain resources such as machinery, raw materials, man hours etc. Some activities use more resources such as labor than others resulting in requirement of more labor to finish those activities. But some activities in the same project require less labor making excess labor redundant having an impact on cost of project. This makes project planning inefficient. Resource leveling helps in smoothening out peaks and valleys in the network diagram i.e. it reallocates resources from over allocated activities (indicated by peaks in network diagram) and to under allocated activities (indicated by valleys in network diagram). This results in much smoother network diagram and proper usage of resources.

8.7.1 ELEMENTS OF RESOURCE LEVELING

The main elements of resource leveling are time used by an activity and resource or number of labor it uses for completion. Hence resource leveling techniques can be broken down into two types:

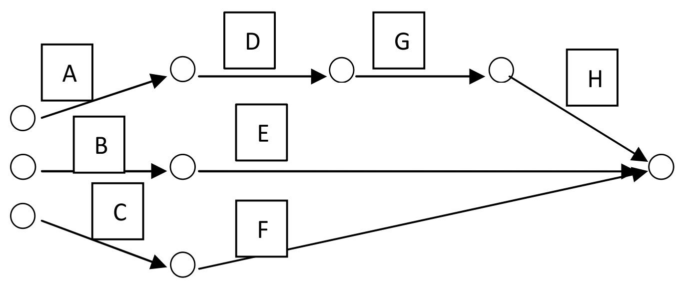
- *Time-limited resource:* if a project manager faces with constraint of finishing project in specified time period then resources of non-critical activities are readjusted according to the slack available with each activity. As it has been discussed with delaying of non-critical activities by their slack project completion time is not impacted so in time constraint projects resources i.e. number of labor are reallocated only for non-critical activities. Activities on the critical path are not adjusted.

- *Labor limited resource:* A project might face a constraint of limited number of workers or labor available to finish off certain activities. In such cases project manager has the liberty to delay the project from its estimated time. This leads to better smoothening i.e. reallocation of resources than in time limited projects. However the first step is always to focus on reallocating of labor involved in carrying out of non-critical activities. If after doing so there are still significant peaks and valleys of resources then critical activities can be altered or delayed.

Example:

Activity	A	B	C	D	E	F	G	H
Predecessor	----	----	----	A	B	C	D	G
Time (days)	4	4	4	2	3	2	3	5
Number of labor	2	1	2	5	2	2	5	3

<u>*Step 1:*</u> Draw the network diagram



<u>*Step 2:*</u> From the network diagram critical path i.e. path that takes maximum time to finish can be identified. There are three paths that can be used to finish the project.

Path 1: A (4) ---D (2) ---G (3) ---H (5) ***= 14 days***

Path 2: B (4) ----E (3) = 7 days

Path 3: C (4) ----F (2) = 6 days

Thus path A---D---G---H is critical path and project will take 14 days to complete.

<u>*Step 3:*</u> On time scale draw a resource diagram which shows number of labor consumed by each activity on a time scale. The values in brackets of each path as shown in following figure indicate number of labor required for a particular activity. So for first day total labor required to carry out activities A, B and C was 5 days. Similarly labor for each day was calculated. It can be seen that days 5 and 6 require maximum labor i.e. 9 and days 10, 11, 12, 13 and 14 require only 3 labors (shown in Fig. 1). Thus, there are significant peaks and valleys.

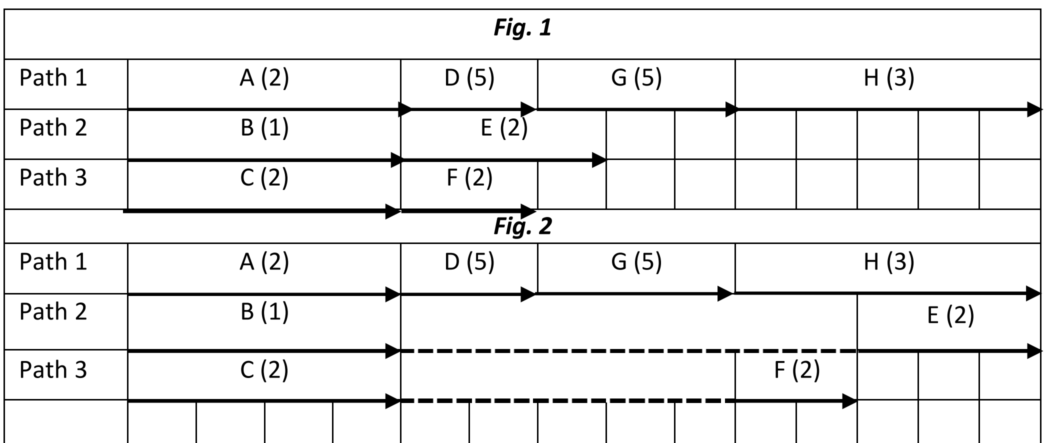
Now if project manager is working under time constraint i.e. project has to be finished in 14 days then first step is to look for delaying non-critical activities by their slack. It was calculated that slack of B, C, E and F activities was 7, 8, 8 and 10 days (as shown in table 1).

Table 1

Activity	Time (days)	EST	LFT	EFT	LST	Slack
A	4	0	4	4	0	0
B	4	0	11	4	7	7
C	4	0	12	4	8	8
D	2	4	6	6	4	0
E	3	3	14	6	11	8
F	2	2	14	4	12	10
G	3	6	9	9	6	0
H	5	9	14	14	9	0

So these can be delayed without delaying the project in order to reallocate resources. There can be number of ways in which this can be done. For illustration we here give one way of doing it.

Delay E by 7 days and F by 5 days. This has been shown in Fig. 2.



Time (days)	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Total labor (Fig. 1)	5	5	5	5	9	9	7	5	5	3	3	3	3	3
Total labor (Fig. 2)	5	5	5	5	5	5	5	5	5	5	5	5	5	5

As shown the labor requirements have smoothened out perfectly as 5 labors are required every day. This is a very simple illustration of resource leveling just to explain the concept and steps involved in the method. Now suppose if project manager faces with a problem of finishing the project within 12 days instead of 14 then that can only be achieved by shortening time of critical activities. This would require applying more resource to such activities and smoothening or leveling of resources becomes difficult.

8.8 SUMMARY

Project is a combination of activities arranged in a sequence to achieve objective of completion of project. Planning of project becomes increasingly difficult if it involves numerous activities. Thus, need of managing project arises which involves arranging activities in a sequence which would help in completing the project most productively. Management of project can be done by using two time oriented network techniques namely PERT and CPM. These techniques are similar to each other in fulfilling objectives of finding project completion time and identifying critical and non-critical activities. They differentiate from each other in a manner that CPM is a deterministic approach used for projects which have been carried out earlier also. Whereas PERT is a probabilistic technique used for finding out probability of completion of project within scheduled time of those projects which are new and project manager has little experience for their conduct. Crashing is a time-cost trade-off technique used to find minimum possible time in which project can be completed with minimal impact on cost.

8.9 GLOSSARY

- ***Project:*** is a non-repetitive combination of activities arranged in a logical sequence.
- ***Program Evaluation and Review Technique (PERT):*** is a probabilistic time oriented network technique used to determine probability of completion of project.
- ***Critical Path Method (CPM):*** is a deterministic time oriented network technique used to determine critical and non-critical activities.
- ***Forward pass:*** is a procedure that involved moving forward in network diagram for determining earliest start and finish times.
- ***Backward pass:*** is a procedure that involved moving backwards in network diagram for determining latest start and finish times.
- ***Crashing:*** is a method of shortening of activity times by adding resources leading to increase in costs.

8.10 ANSWERS TO CHECK YOUR PROGRESS/ SELF ASSESSMENT EXERCISE

Exercise 1

1. Activity-on-Arrow (A-o-A), Activity-on-Node (A-o-N)
2. Activities, logical
3. D
4. D
5. D

Exercise 2

1. Probabilistic
2. Deterministic
3. D
4. B
5. A

8.11 REFERENCES/ SUGGESTED READINGS

- Hillier, F.S. and Lieberman, J. G., *Operations Research,* Tata McGraw Hill, 2009, 10th Reprint.
- Anderson, D.R., Sweeney D. J. and Williams, T.A., *An Introduction to Management Science: Quantitative Approach to Decision Making,* South Western Cengage Learning, 11th Edition.
- Natarajan, A.M., Balasubramani, P. and Tamilarasi, A., *Operations Research,* Pearson, 2012, 9th Edition.

8.12 TERMINAL AND MODEL QUESTIONS

1. A project schedule is as follows:

Activity	1-2	1-3	2-4	3-4	3-5	4-9	5-6	5-7	6-8	7-8	8-10	9-10
Time (days)	4	1	1	1	6	5	4	8	1	2	5	7

   (a) Construct a network diagram.

   (b) Compute total, free and independent float for each activity.

   (c) Find the critical path and project duration.

2. Draw the network diagram and determine various floats for the following project:

Activity	A	B	C	D	E	F
Predecessor	--	--	A	B	A	C,D
Duration	4	9	2	5	6	3

3. Define each of the following terms and indicate how each is determined.
   i. Expected activity time
   ii. Variance of an activity time
   iii. Standard deviation of a path's time.
4. A project has been defined to contain the following activities, along with their time estimates

Activity	**Time estimates (week)**			
	Optimistic	Most likely	Pessimistic	Predecessor
A	1	4	7	-
B	2	6	7	A
C	3	4	6	A,D
D	6	12	14	A
E	3	6	12	D

F	6	8	16	B,C
G	1	5	6	E,F

(a) Draw the critical path diagram

(b) Show the early start, early finish, late start and late finish times.

(c) What is the probability that the project can be completed in 34 weeks?